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Final Report

CONTINUATION OF AN

ADVANCED KLYSTRON STUDY

J. F. Kane and R. N. Wilson

Kane Engineering Laboratories
845 Commercial Street
Palo Alto, California

Contract AF30(602)-2423

MAY 1963

Notices

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Prepared for

Foreword

a. The objective of this program was to develop practical design techniques for broadband, phase stable, high power, klystron microwave amplifiers in a concise mathematical form. The analytic model developed was to be accurate at all signal levels and sufficiently general to include significant space charge modes. The predicted interaction was to be investigated for synthesis methods for the design of amplifiers with specific characteristics. Sample design calculations for an amplifier with supplied parameters was to be carried out following the successful development of the theoretical model.

b. The contractor developed a statement of the beam space charge interaction in the linear or small signal section of the amplifier in matrix notation and from this deduced a gain equation in closed form for the synchronously tuned cavity case. The resulting amplifier design required many cavities tuned to a frequency above the operating frequency. This, however, produced a very high gain peak at the cavity resonant frequency which required elimination to prevent oscillation. The calculations necessary to slightly detune the cavities and eliminate the gain peak were not reducible to any short form so a computer would be required to make the calculations. The large signal beam interaction theory was developed in the infinite beam model for one gap and drift tube, however, the initial conditions encountered at the second gap precluded a solution in closed form for the two or more gap case. The sample design was completed in a very elementary form because of the extremely short time remaining in the contract period when the theoretical portion of the program was completed.

c. The design techniques developed in this contract have been developed in theory only. Experimental klystrons developed by other contractors have demonstrated the validity of some of the results predicted by this theory but no amplifier has been built using this design technique. This large signal theory has eliminated several important approximations made in other theories but requires more work to be completely developed for two or more gap interactions

d. An analytic model of the gain equation for a klystron type amplifier which allows synthesis of amplifiers with broadband and controlled frequency characteristics will be of considerable use in the design of amplifiers for advanced radar systems.

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Abstract

Matrix methods are used in an attempt to clarify the predictions of many resonator klystron response at frequencies above the pass band. The gain expressions have been expressed in factored form in the case of staggered tuning but numerical calculations in the super pass band region have not been completed due to computational length and complexity.

Several subjects in large-signal beam theory are treated. The multigap problem is formulated in terms of an integral equation which is solved within the framework of second order perturbation theory. The disc-model of space-charge forces is used to derive a large-signal equation of motion for a finite diameter beam. Solutions to the equation indicate different space charge reduction factors for each harmonic of the modulating voltage.

Statistical mechanical methods are used to derive equations of motion corresponding to the infinite beam model. Terms appear which cast doubt on the infinite beam model near crossover. The new terms are simulated by the introduction of phenomenological collision loss terms which impede the onset of crossover.

Our results still generally uphold the idea that the klystron can be built as a broadband device with sophisticated phase versus frequency characteristics.

PUBLICATION REVIEW

This report has been reviewed and is approved.

Approved:
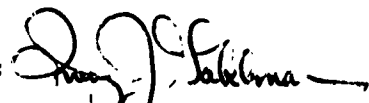
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I. General Introduction

In an earlier engineering report[1] there appeared among other things an analysis of the gain and phase versus frequency characteristic for a many-resonator klystron. The analysis was based upon a single space charge mode-small signal model. In addition, empirical evidence was cited as a basis for prediction of performance of the high signal level gaps and drift tubes for a very broad band klystron having excellent phase linearity in-band and normal klystron efficiency throughout that band.

The analysis consisted of manipulations of well-known space charge relationships. The resulting design called for tuning of the many resonators to resonant frequencies outside of and above the pass band frequencies. This produced a gain characteristic flat and linear in phase within the pass band but it resulted in the possibility of an enormously large gain at one frequency above the pass band. Stagger tuning was recommended as a means of eliminating the large responses above the pass band. While the analysis is believed to be most convincing regarding the pass band frequencies, it leaves something to be desired in terms of conviction at the higher frequencies.

1. RADC-TDR-62-199, J. F. Kane and R. N. Wilson, "Advanced Klystron Study," Kane Engineering Laboratories, Palo Alto, Calif., prepared under Contract AF30(602)-2423 (June 1962) AD-278 223.

Much of the work being reported here was aimed at clari-fying the predictions of tube response at frequencies above the band. Exact expressions have been obtained for this region as well as the pass band region but calculations remain laborious in this region despite very considerable efforts to find means for simplifying them. However, calculation of pass band response is a relatively simple matter for those cases which call for great phase linearity in the pass band.

We have simplified the derivations of formulae here. We find it necessary in this report to resort to a detailed calculation of the tube's response at the frequencies above the pass band as a means of revealing the efficacy of stagger tuning for annihilation of the super pass band response peak. The numerical calculation of gain in the super pass band region has not been completed due to computational length and complexity.

Our work in large-signal theory has progressed in several areas. We have formulated the multigap problem using the infinite beam model for space-charge forces. We obtain an integral equa-tion for the displacement vector which provides a convenient starting point for numerical calculations. We have not been able to obtain closed form large-signal solutions to the multigap problem because of the difficulty with boundary conditions at the second, third, etc., gaps. However, we have solved the integral equation for the case of two gaps, within the framework of second order perturbation theory and this solution displays several interesting features.

We have formulated the large-signal problem using a disc-model for the space-charge forces. The equation of motion takes a more complicated form than that of the infinite beam model. We content ourselves with an approximate large-signal equation of motion which we can solve exactly for certain forms of gap modulating signals. The solutions exhibit space-charge reduction properties and it is shown that a different space-charge reduction factor occurs for each harmonic of the modulating voltage.

We next develop the infinite beam model starting from statistical mechanical foundations. The resulting equations of motion contain heat diffusion terms in addition to the usual terms which occur in the hydrodynamic equations of beam theory. The extra terms appear to become large at crossover and this indicates that the infinite beam model, which pictures the beam as a set of charge planes, may cease to be valid near crossover.

We take a step toward representing the diffusion terms, mentioned above, by introducing a phenomenological collision loss term into the infinite-beam model equation of motion. We are able to solve the resulting equation of motion in closed form in the case of modulation at an infinitesimally extended buncher gap. We find that the presence of the loss tends to impede crossover.

Finally, we remark on some of the directions which future large-signal beam theory might take. There are still many

interesting and difficult problems in large-signal beam theory which remain to be solved. In fact we may claim, so far, only to have scratched the surface in this particular branch of the broad field of non-linear oscillations.

II. N Resonator Gain Formula

We identify a drift tube by assigning to it the number of the resonator which precedes it. The resonators are numbered consecutively from 1 to N starting at resonator number 1, the input resonator. We are interested in the peak values of the a-c current and velocity modulation on the electron beam. These are denoted as follows.

a-c current entering the nth resonator i_n'

a-c current leaving the nth resonator i_n

a-c velocity entering the nth resonator v_n'

a-c velocity leaving the nth resonator v_n

In order to work with dimensionless matrix elements we shall measure a-c velocities in current units as given below.

For the nth gap the following transformation applies.

$$\begin{pmatrix} i_n \\ g v_n \end{pmatrix} = e^{-i\psi_3} \begin{pmatrix} 1 & 0 \\ g z_n & 1 \end{pmatrix} \begin{pmatrix} i_n' \\ g v_n' \end{pmatrix} = (G_n) \begin{pmatrix} i_n' \\ g v_n' \end{pmatrix} \tag{1}$$

where in MKS units

$$g = -\frac{a\omega}{r}\left(\frac{\pi\varepsilon_o I_o}{\eta u_o}\right)^{1/2}$$

$$z_n = M_n^2 \frac{\eta}{u_o} Z_n(\omega) \quad , \quad gz_n \text{ is dimensionless} \qquad (2)$$

ψ_g is an angular measure of time delay in the gap.*

ω is the operating angular frequency

a   is the electron beam's radius

ε_o is the dielectric coefficient of empty space

I_o is the total d-c beam current

η is the specific electronic charge

u_o is the d-c beam velocity

r   is the plasma reduction factor for the beam and drift tube

M_n is the gap coupling coefficient

$Z_n(\omega)$ is the gap impedance at the operating frequency

---

* ψ measures transit time delay in radians of the operating frequency in the structure to which the transformation applies. Due to the linearity of the formalism, only the total phase delay (flight time for beam electrons traversing the entire tube structure) is important. Therefore we have not bothered to attach a phase delay to each tube section. Rather, the total delay is prefixed to the composite tube gain expression at a later point in the analysis.

For the nth drift tube the following transformation applies.

$$\begin{pmatrix} i'_n \\ q\kappa'_n \end{pmatrix} = e^{-j\psi_d} \begin{pmatrix} \cos\theta_n & j\sin\theta_n \\ j\sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} i_{n-1} \\ q\kappa_{n-1} \end{pmatrix} = (D_n) \begin{pmatrix} i_{n-1} \\ q\kappa_{n-1} \end{pmatrix} \quad (3)$$

where $\theta_n = (\omega_q d_n)/u_o$; $\omega_q = r\omega_p$ and ω_p is the beam plasma angular frequency. d_n is the length of the drift tube.

We apply the drift tube and gap transformations in the same order as these are encountered by the electron beam in traversing the tube. The first resonator gap calls for special attention because it is externally driven and is supplied with an unmodulated beam in all cases where the drive power supplied is large compared to available noise energy on the entering beam.

To account for first gap conditions we set

$$\begin{pmatrix} i_1 \\ q\kappa_1 \end{pmatrix} = \begin{pmatrix} 0 \\ q M_1 \left(\dfrac{\eta}{u_o}\right) V_1 \end{pmatrix} \quad (4)$$

This is equivalent to the assumption that a peak r-f voltage V_1 exists on the first gap at all frequencies. Thus the circuits used in applying external drive have been temporarily excluded

as factors in the broadband properties of the rest of the tube.

Now entering the nth resonator we have

$$\begin{pmatrix} i_n' \\ q\nu_n' \end{pmatrix} = e^{-j\psi_t} (D_{n-1})(G_{n-1})(D_{n-2})(G_{n-2}) \cdots (D_2)(G_2)(D_1) \begin{pmatrix} 0 \\ qM_1 \frac{\eta}{u_0} V_1 \end{pmatrix} \quad (5)$$

where $\psi_t = \omega d_{1n}/u_0$ and d_{1n} is the distance between the centers of the 1st and n^{th} gaps.

Noting that the peak r-f voltage on the nth gap is given by

$$V_n = i_n' \, Z_n(\omega) \, M_n \quad (6)$$

we can write

$$(\hat{G}) = \begin{pmatrix} V_n/V_1 \\ q \dfrac{V_n}{V_1} \dfrac{\nu_n'}{i_n'} \end{pmatrix}$$

$$= K \, (D_{n-1})(G_{n-1})(D_{n-2})(G_{n-2}) \cdots (D_2)(G_2) \begin{pmatrix} j\sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \quad (7)$$

where $K = e^{-j\psi_t} g M_{1u} \frac{\eta}{u_o} M_n Z_n(\omega)$. V_n/V_1 is of course the gain. The relatively complicated quantity representing the modified velocity is of no immediate interest in a gain calculation.

The matrix (D_n) has eigenvalues λ_1, $\lambda_2 = e^{\pm j\theta_n}$ and is diagonalized by U where

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad (8)$$

with

$$V = U^{-1} \, , \quad U U^{-1} = I \qquad (9)$$

We write $(\hat{G})$ in the form

$$(\hat{G}) = K U^2 (D_{n-1}) U^2 (G_{n-1}) U^2 (D_{n-2}) \cdots U^2 (D_2) U^2 (G_2) U^2 \begin{pmatrix} j \sin \theta_1 \\ \cos \theta_1 \end{pmatrix} \qquad (10)$$

and make use of the following definitions.

$$U(D_n)U = \begin{pmatrix} e^{j\theta_n} & 0 \\ 0 & e^{-j\theta_n} \end{pmatrix} \equiv (\Theta_n) \qquad (11)$$

$$U\begin{pmatrix} j\sin\theta_1 \\ \cos\theta_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{j\theta_1} \\ -e^{-j\theta_1} \end{pmatrix} \equiv \frac{1}{\sqrt{2}}(\hat{\Theta}_1) \qquad (12)$$

$$U(G_n)U = \begin{pmatrix} 1+\gamma_n & \gamma_n \\ -\gamma_n & 1-\gamma_n \end{pmatrix} \equiv \Gamma_n \qquad (13)$$

$$\gamma_n = \frac{1}{2} g z_n \qquad (14)$$

The vector $(\hat{G})$ now becomes

$$(\hat{G}) = \frac{1}{\sqrt{2}} k \upsilon \Theta_{n-1} \Gamma_{n-1} \Theta_{n-2} \Gamma_{n-2} \cdots \Theta_{2} \Gamma_{2} (\hat{\theta}_{1}) \tag{15}$$

If we specialize the tube design by setting all n drift tube lengths equal we have

$$(\hat{G}) = \frac{1}{\sqrt{2}} k \upsilon \prod_{i=2}^{N-1} (\Theta \Gamma_{i}) (\hat{\theta}) \tag{16}$$

We notice that

$$(\Theta \Gamma_{i}) (\hat{\theta}) = (2 \cosh u_{i}) (\hat{\theta}) + (\hat{1}) \tag{17}$$

where

$$(\hat{1}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\cosh u \equiv \cos \theta + \tfrac{1}{2} j g z \sin \theta \tag{18}$$

This is the same notation that was used in the previous engineering report.[2]

In addition, we observe that

$$\Theta \Gamma_{2} (\hat{1}) = - (\hat{\theta}) \tag{19}$$

2. RADC-TDR-62-199.

Multiplying the first few matrices on the right of the gain expression into the vector θ we get

$$\Theta \, \Gamma_2 \, \hat{\theta} = 2\cosh u_2 (\hat{\theta}) + (\hat{1})$$

(20)

$$\Theta \, \Gamma_3 \Theta \, \Gamma_2 \, \hat{\theta} = (2\cosh u_3 \, 2\cosh u_2 - 1)(\hat{\theta}) + 2\cosh u_2 (\hat{1})$$

(21)

$$\Theta \, \Gamma_4 \cdots \Theta \, \Gamma_2 \, \hat{\theta} = [\, 2\cosh u_4 (2\cosh u_3 \, 2\cosh u_2 - 1)$$
$$- 2\cosh u_2 \,] \, \hat{\theta}$$
$$+ (2\cosh u_3 \, 2\cosh u_2 - 1)\, \hat{1}$$

(22)

etc. Observation of the above polynomials and the manner in which they are generated shows that the gain can be written

$$\frac{V_n}{V_1} = j K \sin \theta \, |A_{N-1}|$$

(23)

where

$$|A_i| = \begin{vmatrix} 2\cosh u_i & 1 & & & O \\ 1 & 2\cosh u_{i-1} & 1 & & \\ & 1 & \ddots & & \\ & & & 2\cosh u_3 & 1 \\ O & & & 1 & 2\cosh u_2 \end{vmatrix}$$

(24)

which determinant is (N - 2) by (N - 2).

If there is no stagger tuning the u_i are identical and the gain becomes

$$\frac{V_n}{V_1} = jk \sin\theta \; \frac{\sinh(N-1)u}{\sinh u} \tag{25}$$

This result we have seen before.

III. Computation

We have searched for simple means for evaluating the determinant $|A_i|$ such that a tube design engineer could perform design calculations at a desk in a few hours. We have not as yet succeeded in this. To perform the calculations for a tube of four or more low level resonators is still a very laborious undertaking without a computer. Computer programs exist, however, which would permit stagger tuned pass band computations in a few minutes. In consequence of this, we have developed the following technique for desk computation.

The gain determinant $|\Delta_i|$, when written out, appears as the quotient of two polynomials in the modulating frequency. We wish to express these two polynomials in factored form for the case of non-synchronous tuning. Since the factorization is somewhat involved we first present the general results. It is to be noted that the factorization is performed separately for the numerator and denominator, thus avoiding the pitfalls of perturbation theory in its usual sense.

At the end we present a few special cases.

Results

We express the gain determinant, given by Eq. (39) below, in factored form when the cavities have resonant frequencies

$$
\left.
\begin{aligned}
\omega_k &= \omega_0 + \Delta \omega_{0k} \\
\delta_k &= \frac{\Delta \omega_{0k}}{\omega_c} \ll 1
\end{aligned}
\right\}
\tag{26}
$$

and we allow a change in drift tube lengths

$$
\theta \longrightarrow \theta_e + \Delta \theta .
\tag{27}
$$

For an $(N - 1)$-dimensional determinant we obtain

$$
\Delta = \frac{\Delta \mathcal{H}}{\Delta \theta}
\tag{28}
$$

where

$$
\Delta \theta = \prod_{\ell=1}^{N-1} (\omega - c_0 - \omega_0 \delta_\ell)(\omega - d_0 + \omega_0 \delta_\ell)
\tag{29}
$$

$$
\Delta \mathcal{H} = \prod_{\ell=1}^{N-1} \left\{ \left[N_c + \frac{1}{N} \sum_{j=1}^{N-1} N_j \right] + 2 \left[D_o + \frac{1}{N} \sum_{j=1}^{N-1} D_j \right] \cos \frac{\pi \ell}{N} \right.
$$

$$
\left. - \frac{1}{N} \sum_{j=1}^{N-1} \cos \frac{2\pi \ell j}{N} \left(N_j + 2 D_j \cos \frac{2\pi \ell}{N} \right) \right\}
\tag{30}
$$

$$N_o = K_o a_o (\omega - a_o)(\omega - b_o) \tag{31}$$

$$N_j = -K_o a_o (\omega - a_o)(\omega - b_o)(\epsilon + \tan \theta \, \Delta \theta)$$
$$-K_o a_o [\Delta_{1j}(\omega - b_o) + \Delta_{2j}(\omega - a_o)] \tag{32}$$

$$D_o = (\omega - c_o)(\omega - d_o) \tag{33}$$

$$D_j = \omega_o (d_o - c_o)\delta_j \tag{34}$$

$$K_o = 2 \cos \theta_o \tag{35}$$

$$a_o = 1 - \frac{g'}{2\Omega c} \tan \theta_o \tag{36}$$

$$b = j\frac{G}{c} \tag{37}$$

$$\begin{Bmatrix} a_o \\ b_o \end{Bmatrix} = j \frac{\omega_o'}{2Q_1} \pm \omega_o' \sqrt{1 - \frac{1}{4q_1^2}} \, , \, \begin{Bmatrix} c_o \\ d_o \end{Bmatrix} = j \frac{\omega_o}{2q_o} \pm \omega_o \tag{38}$$

ω_o, ω_o', Q_o, Q_o' are given by Eq. (48) below.

Factorization

We begin by rewriting the gain determinant, given by Eq. (24), in the following form. Note that N of this section is to be identified with N - 1 of Section II.

$$\Delta = \begin{vmatrix} 2\cosh u_1 & 1 & & & \\ 1 & 2\cosh u_2 & & & \bigcirc \\ & & \diagdown & & \\ & \bigcirc & & & 1 \\ & & & 1 & 2\cosh u_{N-1} \end{vmatrix} \quad (39)$$

This is accomplished by some interchanging of rows and columns in Eq. (24). We examine cosh u

$$2\cosh u_\ell = 2\cos\theta + j\,\frac{g\sin\theta}{G + j\left(\omega C_\ell - \frac{1}{\omega L_\ell}\right)} \Bigg\}$$

$$g = -\omega\,\frac{a}{r}\left(\frac{\pi\epsilon_0 I_0}{\eta u_0}\right)^{1/2}\frac{\eta}{u_0}M^2 \quad (40)$$

We wish to express Δ in factored form when $u_1 \neq u_2 \neq u_3 \neq \ldots \neq u_{N-1}$. We assume that the only frequency dependence of g is in the multiplicative ω term. Define

$$g = -\left(\frac{\omega}{\Omega}\right)g', \quad g' = \frac{a\Omega}{r}\left(\frac{\pi\epsilon_0 I_0}{\eta u_0}\right)^{1/2}\frac{\eta}{u_0}M^2 \quad (41)$$

where Ω is a constant frequency chosen equal to ω_o, the resonant frequency of the cavities in the synchronous case.

$$\omega_o^2 = \frac{1}{LC} \quad (42)$$

Ω may be given any value. It occurs only for dimensional reasons.

Write $2 \cosh u_0$ in factored form. The synchronous, $2 \cosh u_0$, is

$$2 \cosh u_0 = 2 \cos \theta_0 \, \frac{\left[\omega^2\left[1 - \frac{g'}{2\Omega c} \tan \theta\right] - \omega\left(\frac{jG}{c}\right) - \omega_0^2\right]}{\left[\omega^2 - \omega\left(\frac{jG}{c}\right) - \omega_0^2\right]} \tag{43}$$

as can easily be verified from Eq. (40). Introduce the notation

$$2 \cosh u_0 = k\left(\frac{\alpha\omega^2 - b\omega - \omega_0^2}{\omega^2 - b\omega - \omega_0^2}\right) \tag{44}$$

where

$$\left.\begin{aligned} K &= 2 \cos \theta \\[6pt] \alpha &= 1 - \frac{g'}{2\Omega c} \tan \theta \\[6pt] b &= \frac{jG}{c} \end{aligned}\right\} \tag{45}$$

Factoring Eq. (43) we can write

$$2 \cosh u_0 = K\alpha \, \frac{(\omega - a_0)(\omega - b_0)}{(\omega - c_0)(\omega - d_0)} \quad, \tag{46}$$

where

$$\left\{\begin{aligned} a_0 \\ b_0 \end{aligned}\right\} = j\frac{\omega_0'}{2q_1} \pm \omega_0'\sqrt{1 - \frac{1}{4q_1^2}}, \quad \left\{\begin{aligned} c_0 \\ d_0 \end{aligned}\right\} = j\frac{\omega_0}{2q_0} \pm \omega_0\sqrt{1 - \frac{1}{4q_0^2}} \tag{47}$$

$$\omega_o = \frac{1}{LC} \quad , \quad \omega_o' = \frac{\omega_o}{\left(1 - \frac{g'}{2\Omega c} \tan \theta\right)^{1/2}} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}$$

$$\frac{1}{Q_o} = \frac{G}{C\omega_o} \quad , \quad \frac{1}{Q_1} = \frac{1}{Q_o} \frac{1}{\left(1 - \frac{g'}{2\Omega c} \tan \theta\right)^{1/2}} \tag{48}$$

We consider the case where

$$1 \gg \frac{1}{4 Q_o^2} \tag{49}$$

Then

$$\left\{\begin{array}{c} a_o \\ b_o \end{array}\right\} = j \frac{\omega_o'}{2 Q_1} \pm \omega_o' \sqrt{1 - \frac{1}{4 Q_1^2}} \quad , \quad \left\{\begin{array}{c} c_o \\ d_o \end{array}\right\} = j \frac{\omega_o}{2 Q_o} \pm \omega_o \tag{50}$$

where we have allowed $1/(4Q_1^{\ 2})$ to be comparable with unity.

To treat the non-synchronous case we start with the synchronous case and consider the effect of changing the resonant frequency of the jth cell. We also allow a change in drift tube length. Eq. (39) is only valid when all drift lengths are equal. Hence we change all the drift tube lengths by the same amount. We let

$$\omega_o \longrightarrow \omega_o + \Delta \omega_o j \longrightarrow \omega_o (1 + \delta_j) \tag{51}$$

$$S_j = \frac{\Delta \omega_{oj}}{\omega_o} << 1 \qquad (52)$$

$$\theta \longrightarrow \theta_o + \Delta\theta \quad , \quad \Delta\theta << 1 \qquad (53)$$

Then keeping only lowest order terms

$$\left\{ \begin{array}{c} a_o \\ b_o \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} a_j \\ b_j \end{array} \right\}$$

$$\left\{ \begin{array}{c} a_j \\ b_j \end{array} \right\} = \left\{ \begin{array}{c} a_o \\ b_o \end{array} \right\} + \left\{ \begin{array}{c} \Delta_{1j} \\ \Delta_{2j} \end{array} \right\} \qquad (54)$$

$$\left\{ \begin{array}{c} \Delta_{1j} \\ \Delta_{2j} \end{array} \right\} = j \frac{\omega_o'}{2q_1} \epsilon \pm \left\{ \frac{\omega_o' S_j'}{\sqrt{1-\frac{1}{4q_1^2}}} + \frac{\omega_o'(1-\frac{1}{2q_1^2})\epsilon}{2\sqrt{1-\frac{1}{4q_1^2}}} \right\} \qquad (55)$$

where

$$\epsilon = \frac{\frac{g'}{2\Omega c}\sec^2\theta_o \, \Delta\theta}{1 - \frac{g'}{2\Omega c}\tan\theta_o} << 1 \qquad (56)$$

$$\left\{ \begin{array}{c} c_0 \\ d_0 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} c_j \\ d_j \end{array} \right\} \tag{57}$$

$$\left\{ \begin{array}{c} c_j \\ d_j \end{array} \right\} = \left\{ \begin{array}{c} c_0 \\ d_0 \end{array} \right\} \pm \omega_c \, \delta_j \tag{58}$$

Eqs. (54), (58) tell us how the zeros and poles of Eq. (46) move when we change the resonant frequency of the cavity and the drift tube length associated with cell j.

To get the perturbed $2 \cosh u_0$ we only have to put in the changes in K, α.

$$K = 2 \cot \theta \longrightarrow 2 \cot(\theta + \Delta\theta) \simeq 2 \cot\theta - 2 \sin\theta \, \Delta\theta$$
$$= K_0 \left\{ 1 - \tan\theta \, \Delta\theta \right\} \tag{59}$$

$$\alpha \longrightarrow 1 - \frac{g'}{2\Omega c} \tan\theta_0 - \frac{g' \sec^2\theta_0}{2\Omega c} \Delta\theta \tag{60}$$

$$\alpha = \left(1 - \frac{g'}{2\Omega c} \tan\theta_0 \right) \left\{ 1 - \frac{\frac{g'}{2\Omega c} \sec^2\theta_0 \, \Delta\theta}{1 - \frac{g'}{2\Omega c} \tan\theta_0} \right\} \tag{61}$$

$$= \alpha_0 \, (1 - \epsilon)$$

With Eqs. (54), (58), (59), (61) we obtain $2 \cosh u_j$.

$$2 \cosh u_j = K_o V_o (1-\epsilon)(1-\tan\theta\Delta\theta) \frac{(\omega-a_o-\Delta_{1j})(\omega-b_o-\Delta_{2j})}{(\omega-c_o-\omega_o S_j)(\omega-d_o+\omega_o S_j)} \quad (62)$$

Note that we have treated the movement of the zeros and poles separately and they are constrained in their movement by Eqs. (51), (52), (55), (56), (61) in order that the laws of physics remain satisfied. That is, we allow only move-ments of the poles and zeros which are physically realizable. Whether the chosen movements are practically realizable is another question.

Now we turn to the problem of evaluating the gain deter-minant given by Eq. (39). First we note the following properties of a special matrix and its determinant. Consider the matrix

$$(63)$$

The eigenvalues of $\mathcal{M}$ are

$$\lambda_i = \gamma + 2a \cos \frac{\pi i}{N} \tag{64}$$

The matrix which diagonalizes $\mathcal{M}$ is

$$U = U^{-1} = \left\{ \sqrt{\frac{2}{N}} \sin \frac{\ell m \pi}{N} \right\}; \quad \ell, m = 1, 2, ---, N-1 \tag{65}$$

We now write Eq. (39) with the aid of Eq. (62)

$$\Delta = \frac{\Delta \mathcal{K}}{\Delta \mathcal{D}} \tag{66}$$

where

$$\Delta \mathcal{D} = \prod_{\ell=1}^{N-1} (\omega - c_o - \omega_o S_\ell)(\omega - d_o + \omega_o S_\ell) \tag{67}$$

$$\Delta \mathcal{H} = \begin{vmatrix} N_o + N_1 & D_o + D_1 & & & & \\ D_o + D_2 & N_o + N_2 & D_o + D_2 & & O & \\ & D + D_3 & \diagdown & \diagdown & & \\ & & \diagdown & \diagdown & & \\ & O & & \diagdown & \diagdown & D_o + D_{N-2} \\ & & & & D_o + D_{N-1} & N_o + N_{N-1} \end{vmatrix} \tag{68}$$

where

$$D_o + D_j = (\omega - c_o - \omega_o \delta_j)(\omega - d_o + \omega_o \delta_j) \tag{69}$$

$$N_o + N_j = K_o a_o (1 - \epsilon)(1 - \tan\theta \, \Delta\, \theta) \cdot$$
$$\cdot (\omega - a_o - \Delta_{1j})(\omega - b_o - \Delta_{2j}) \tag{70}$$

Note that we have calculated the effect of tuning separately for the numerator and denominator of Δ. That is, we calculate the movement of the poles and zeros separately and in this sense the treatment differs from perturbation theory. Perturbation theory fails when we must manipulate resonant quantities; e.g., move poles and zeros of a gain determinant.

The denominator of Δ given by Eq. (67) appears in factored form. It remains to factor the numerator, $\Delta\eta$, given by Eq. (68). We factor $\Delta\eta$ through first order in the tuning. Rewrite η in the form

$$\eta = \eta_0 + \eta_1 , \tag{71}$$

where

$$\eta_0 = \left\{ \begin{array}{ccccc} N_0 & D_0 & & & O \\ D_0 & N_0 & D_0 & & \\ & & \ddots & & \\ & & & & D_0 \\ O & & & D_0 & N_0 \end{array} \right\} , \quad \eta_1 = \left\{ \begin{array}{ccccc} N_1 & D_1 & & & O \\ D_2 & N_2 & D_3 & & \\ & & \ddots & & \\ & & & & D_{N-1} \\ O & & & D_{N-1} & N_{N-1} \end{array} \right\} \tag{72}$$

$$N_o = K_o a_o (\omega - a_o)(\omega - b_o)$$

$$N_j = -K_o a_o (\omega - a_o)(\omega - b_o)(\epsilon + \tan \theta \, \Delta \, \theta)$$

$$-K_o a_o [\Delta_{1j}(\omega - b_o) + \Delta_{2j}(\omega - a_o)] \qquad (73)$$

$$D_o = (\omega - c_o)(\omega - d_o)$$

$$D_j = \omega_o (d_o - c_o)\delta_j$$

We transform η with the matrix U given by Eq. (65) and note that

$$\Delta \{U^{-1} \eta U\} = (\Delta U^{-1})(\Delta \eta)(\Delta U) = \Delta \eta \qquad (74)$$

The result is

$$\Delta \eta = \Delta \{\eta_o' + \underbrace{U^{-1} \eta_1 U}_{\{\eta_{ij}'\}}\} \qquad (75)$$

where

$$\eta_o' = \left\{ \begin{matrix} \lambda_1 & \lambda_2 & & O \\ & & \ddots & \\ O & & & \lambda_{N-1} \end{matrix} \right\} \qquad (76)$$

$$\lambda_i = N_0 + 2 D_0 \cot \frac{\pi i}{N} \quad , \quad i = 1, 2, \cdots , N-1 \tag{77}$$

$$\Delta \chi_0 = \Delta \chi_0' = \prod_{\ell=1}^{N-1} \left\{ N_0 + 2 D_0 \cot \frac{\pi \ell}{N} \right\} \tag{78}$$

Also

$$\eta' = \left\{ \begin{array}{cccc} \lambda_1 + \eta_{11}', \ \eta_{12}', & \text{---} \ \cdots \ \text{---} & \eta_{1,N-1}' \\ \eta_{11}', \ \lambda_2 + \eta_{22}', & \text{--- --- ---} & \eta_{2,N-1}' \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \eta_{N-1,1}' & \text{--- --- ---} & \lambda_{N-1} + \eta_{N-1,N-1}' \end{array} \right\} \tag{79}$$

We wish to calculate $\Delta \chi'$ through first order in the quantities η'_{ij} . To see what this is, consider expanding $\Delta \chi'$ by minors along the first row.

$$\Delta \eta' = (\lambda_1 + \eta_{11}') \left| \begin{array}{cccc} \lambda_2 + \eta_{22}', \ \eta_{23}', & \text{--- --- ---} & \eta_{2,N-1}' \\ \eta_{32}', \ \lambda_3 + \eta_{33}', & \text{--- --- ---} & \eta_{3,N-1}' \\ \vdots & \vdots & \vdots \\ \eta_{N-1,1}' & \text{--- --- ---} & \lambda_{N-1} + \eta_{N-1,N-1}' \end{array} \right| + \text{(next page)}$$

$$-\eta'_{12} \begin{vmatrix} \eta'_{21} & \eta'_{23} & \eta'_{24} & \cdots & \eta'_{2,N-1} \\ \eta'_{31} & \lambda'_3 + \eta'_{33} & \eta'_{34} & \cdots & \eta'_{3,N-1} \\ \eta'_{41} & \eta'_{43} & \lambda'_4 + \eta'_{44} & \cdots & \eta'_{4,N-1} \\ \vdots & & & & \\ \eta'_{N-1,1} & \eta'_{N-1,3} & \cdots & & \lambda_{N-1} + \eta'_{N-1,N-1} \end{vmatrix} \tag{80}$$

$$+ \quad - \; - \; - \; - \; - \; - \; - \; -$$

All terms in (80) except the first are second order in the
η'_{ij} and therefore are negligible in our approximation.
Through first order the first term is

$$\Delta\eta' = \lambda_1 \begin{vmatrix} \lambda_2 + \eta'_{22} & \eta'_{23} & \cdots & \eta'_{2,N-1} \\ \eta'_{32} & \lambda_3 + \eta'_{33} & \cdots & \eta'_{3,N-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \eta'_{N-1,2} & \cdots & & \lambda_{N-1} + \eta'_{N-1,N-1} \end{vmatrix} + \eta'_{11} \prod_{\ell=2}^{N-1} \lambda_\ell \tag{81}$$

Now the determinant multiplying λ_1 in Eq. (81) is the same
form as our original determinant $\Delta\eta'$ except of dimension
(N - 2) x (N - 2). Hence through first order

$$\Delta \varkappa' = \lambda_1 (\lambda_2 + \varkappa_{12}') \begin{vmatrix} \lambda_3 + \varkappa_{33}' & \varkappa_{34}' & ----- & \varkappa_{3,N-1}' \\ \varkappa_{43}' & \lambda_4 + \varkappa_{44}' & ----- & \varkappa_{4,N-1}' \\ \vdots & & & \vdots \\ \varkappa_{N-1,3}' & ------- & & \lambda_{N-1} + \varkappa_{N-1,N-1}' \end{vmatrix}$$

$$+ \mathcal{O}(\varkappa_{ij}')^2 + \varkappa_{11}' \prod_{\ell=2}^{N-1} \lambda_\ell \tag{82}$$

$$= \lambda_1 \lambda_2 \begin{vmatrix} \lambda_3 + \varkappa_{33}' & \varkappa_{34}' & ------- & \varkappa_{3,N-1}' \\ \varkappa_{43}' & \lambda_4 + \varkappa_{44}' & ------ & \varkappa_{4,N-1}' \\ \vdots & & & \vdots \\ \varkappa_{N-1,3}' & ----------- & & \lambda_{N-1} + \varkappa_{N-1,N-1}' \end{vmatrix}$$

$$+ \varkappa_{11}' \prod_{\ell=2}^{N-1} \lambda_\ell + \lambda_1 \varkappa_{22}' \prod_{\ell=3}^{N-1} \lambda_\ell \tag{83}$$

We obtain immediately by induction that, through first order in the η'_{ij} ,

$$\Delta \chi = \prod_{\ell=1}^{N-1} \lambda_\ell + \sum_{\ell=1}^{N-1} \eta'_{\ell\ell} \prod_{j\neq\ell} \lambda_\ell \tag{84}$$

Eq. (84) we express in the form

$$\Delta \chi = \prod_{\ell=1}^{N-1} \{\lambda_\ell + \eta'_{\ell\ell}\} . \tag{85}$$

Thus what is left to do is calculate $\eta'_{\ell\ell}$

$$\eta'_{\ell\ell} = (U\chi,U)_{\ell\ell} \tag{86}$$

We use Eq. (65) and (72) to obtain

$$\eta'_{\ell\ell} = U_{\ell 1}(N_1 U_{1\ell} + D_1 U_{2\ell})$$
$$+ \sum_{j=2}^{N-2} U_{\ell j}(D_j U_{j-1,\ell} + N_j U_{j\ell} + D_j U_{j+1,\ell})$$
$$+ U_{\ell N-1}(D_{N-1} U_{N-2,\ell} + N_{N-1} U_{N-1,\ell}). \tag{87}$$

Inserting the values of U_{ij}, Eq. (87) becomes

$$\eta'_{\ell\ell} = \frac{1}{N} \sum_{j=1}^{N-1} \left(1 - \cos\frac{2\pi\ell j}{N}\right)\left(N_j + 2D_j \cos\frac{\pi\ell}{N}\right) \quad (88)$$

Combining Eq. (85) with (88) the result is

$$\Delta\eta = \prod_{\ell=1}^{N-1} \left\{ N_0 + 2D_0 \cos\frac{\pi\ell}{N} \right.$$

$$\left. + \frac{1}{N}\sum_{j=1}^{N-1}\left(1 - \cos\frac{2\pi\ell j}{N}\right)\left(N_j + 2D_j\cos\frac{\pi\ell}{N}\right)\right\} \quad (89)$$

or

$$\Delta\eta = \prod_{\ell=1}^{N-1}\left\{\left[N_0 + \frac{1}{N}\sum_{j=1}^{N-1}N_j\right] + 2\left[D_0 + \frac{1}{N}\sum_j D_j\right]\cos\frac{\pi\ell}{N}\right.$$

$$(90)$$

$$\left. - \frac{1}{N}\sum_{j=1}^{N-1}\cos\frac{2\pi\ell j}{N}\left(N_j + 2D_j\cos\frac{\pi\ell}{N}\right)\right\}$$

From Eqs. (73), (55) and (50)

$$N_j = -k_o \alpha_o (\omega - a_o)(\omega - b_o)(\epsilon + \tan \theta_o \Delta \theta)$$

$$- k_o \alpha_o \left[j\omega \frac{\omega_o'}{Q_1} + 2\left(\frac{\omega_o'}{2Q_1}\right)^2 + (\omega_o')^2 \left(1 - \frac{1}{2Q_1^2}\right) \right] \epsilon$$

$$- k_o \alpha_o \, 2(\omega_o')^2 S_j \qquad (91)$$

$$D_j = \omega_o (d_o - c_o) S_j$$

Substituting in Eq. (90) we obtain

$$\Delta \eta = \prod_{\ell=1}^{N-1} \eta_\ell \qquad (92)$$

where

$$\eta_\ell = k_o \alpha_o \left[1 - \frac{(N-2)}{N}(\epsilon + \tan \theta_o \Delta \theta) \right](\omega - a_o)(\omega - b_o)$$

$$+ 2 \cos \frac{\pi \ell}{N} (\omega - c_o)(\omega - d_o)$$

$$- \frac{(N-2)}{N} k_o \alpha_o \left[j\omega \frac{\omega_o'}{Q_1} + 2\left(\frac{\omega_o'}{2Q_1}\right)^2 + (\omega_o')^2 \left(1 - \frac{1}{2Q_1^2}\right) \right] \epsilon \qquad (93)$$

$$- \frac{2 k_o \alpha_o}{N} (\omega_o')^2 \sum_k \left(1 - \cos \frac{2\pi \ell k}{N}\right) S_k$$

$$+ 2 \frac{\omega_o}{N} (d_o - c_o) \cos \frac{\pi \ell}{N} \sum_k \left(1 - \cos \frac{2\pi \ell k}{N}\right) S_k$$

Eq. (93) is a quadratic in ω which is easily factored.

Special Cases

We look at the following special cases.

(1) $\qquad \dfrac{g'}{2\Omega c}\tan\theta_0 \ll 1$ $\left.\begin{array}{l}\\[2em]\end{array}\right\}$ $a_0 \simeq c_0$

$\qquad \dfrac{g'}{2\Omega c}\sec^2\theta_0\,\Delta\theta \ll 1$ $\left.\begin{array}{l}\\\end{array}\right\}$ $b_0 \simeq d_0$

We consider ε negligible and Eq. (93) becomes

$$\eta_\ell = k_0 \alpha_0 \left[1 - \frac{(N-2)}{N}\tan\theta_0\,\Delta\theta\right](\omega - a_0)(\omega - b_0)$$

$$+ \, 2\cos\frac{\pi\ell}{N}\,(\omega - a_0)(\omega - b_0)$$

$$- \, \frac{2k_0\alpha_0}{N}(\omega_0)^2 \sum_h \left(1 - \cos\frac{2\pi\ell h}{N}\right)\delta_h \qquad (94)$$

$$+ \, 2\frac{\omega_0}{N}(d_0 - c_0)\cos\frac{\pi\ell}{N}\sum_h \left(1 - \cos\frac{2\pi\ell h}{N}\right)\delta_h$$

In this approximation the gain determinant (39) is independent of frequency when $\delta_k = 0$.

$$d_0 - c_0 = -2\omega_0 \qquad (95)$$

$$\eta_\ell = \left\{k_0\alpha_0\left(1 - \frac{N-2}{N}\tan\theta_0\,\Delta\theta\right) + 2\cos\frac{\pi\ell}{N}\right\}(\omega - a_0)(\omega - b_0)$$

$$- \, \frac{2\omega_0^2}{N}\left(k_0\alpha_0 + 2\cos\frac{\pi\ell}{N}\right)\sum_h\left(1 - \cos\frac{2\pi\ell h}{N}\right)\delta_h \qquad (96)$$

(2) In addition to the conditions of (1) we let $\sum_k \delta_k = 0$.
Then

$$
\eta_\ell = \left\{ k_0 \alpha_0 \left(1 - \frac{(N-1)}{N} \tan \theta_0 \, \Delta\theta \right) + 2 \cos \frac{\pi\ell}{N} \right\} (\omega - a_0)(\omega - b_0)
$$

$$
+ \frac{2\omega_0^2}{N} \left(k_0 \alpha_0 + 2 \cos \frac{\pi\ell}{N} \right) \sum_k \cos \frac{2\pi\ell h}{N} \, \delta_h \tag{97}
$$

(3) Same conditions as (2) but with $\Delta\theta = 0$.

$$
\eta_\ell = \left(k_0 \alpha_0 + 2 \cos \frac{\pi\ell}{N} \right) \left\{ (\omega - a_0)(\omega - b_0) \right.
$$

$$
\left. + \frac{2\omega_0^2}{N} \sum_k \cos \frac{2\pi\ell h}{N} \, \delta_h \right\} \tag{98}
$$

The special case $\delta_k = (-1)^k \delta$ requires $N - 1$ be an even
integer, if

$$
\sum_h \delta_h = 0 \tag{99}
$$

is to be satisfied. However, we find

$$
\sum_{h=1}^{N-1} \cos \frac{2\pi\ell h}{N} (-1)^h \delta = 0 \tag{100}
$$

in this case. Hence, to the order of our approximation,
this type of stagger tuning has no effect on the numerator

of the gain determinant. It makes the poles of order (N - 1)/2
instead of (N - 1).

(4) In addition to the conditions of (1) we let

$$S_1 = S_2 = --- = S_{N-1} = S \tag{101}$$

$$S \sum_{h} \left(1 - \cos \frac{2\pi \ell h}{N}\right) = (N-1)S + S = NS \tag{102}$$

Hence

$$\eta_\ell = \left\{ k_0 \alpha_0 \left(1 - \frac{N-2}{N} \tan \Theta_0 \Delta \Theta\right) + 2\cos \frac{\pi \ell}{N} \right\} (\omega - a_0)(\omega - b_0)$$

$$- 2\omega_0^2 \left(k_0 \alpha_0 + 2\cos \frac{\pi \ell}{N} \right) S \tag{103}$$

If, in addition, $\Delta \Theta = 0$

$$M_{\ell} = \left(k_0 \alpha_0 + 2\cos \frac{\pi \ell}{N} \right) \left[(\omega - a_0)(\omega - b_0) - 2 S \omega_0^2 \right] \tag{104}$$

In this case

$$\Delta = \frac{\left\{ \prod_{\ell=1}^{N-1} \left(k_0 \alpha_0 + 2 \cos \frac{\pi \ell}{N} \right) \right\} \left[(\omega - a_0)(\omega - b_0) - 2\omega_0^2 \delta \right]^{N-1}}{(\omega - a_0 - \omega_0 \delta)^{N-1} (\omega - b_0 + \omega_0 \delta)^{N-1}} \tag{105}$$

Hence this type of tuning does not remove the confluence of the poles.

From (3) and (4) it is clear that we should make all δ's different. For example:

(5) In addition to the conditions of (1) we let $\Delta \theta = 0$ and

$$\delta_k = k (-1)^k \delta \tag{106}$$

This will make all poles and zeros move in first order.

IV. Numerical Computations

At this point we have not completed any numerical com-
putations which are based on stagger tuned tubes. As a result
we are not yet in a position to demonstrate by actual computation
that the so-called super pass band gain peak can be annihiliated
or reduced to a safely small maximum value. On an intuitive
basis we do not question that this gain peak can be annihiliated
in practical structures. For the design of broadband structures
with gain and phase linearity as required, we use the degen-
erately tuned gain formula which applies in the region below
the resonant frequencies of the resonators.

Figure 1 shows the real and imaginary parts of cosh u as
a function of frequency. This represents a tube having 10 mega-
watts of output capability using a beam microperveance of unity
near 2800 megacycles per second for which $\delta = 0$. The range
of frequencies in which the real part of cosh u is greater than
unity and within which the imaginary part of cosh u is negligible
is of the greatest interest here. So long as the imaginary part
of cosh u is small, there will be at most a small and monotonic
departure of the phase versus frequency curve from a strictly
linear curve.

Figure 2 shows the gains which can be expected from
klystrons whose small signal level resonators are numerous

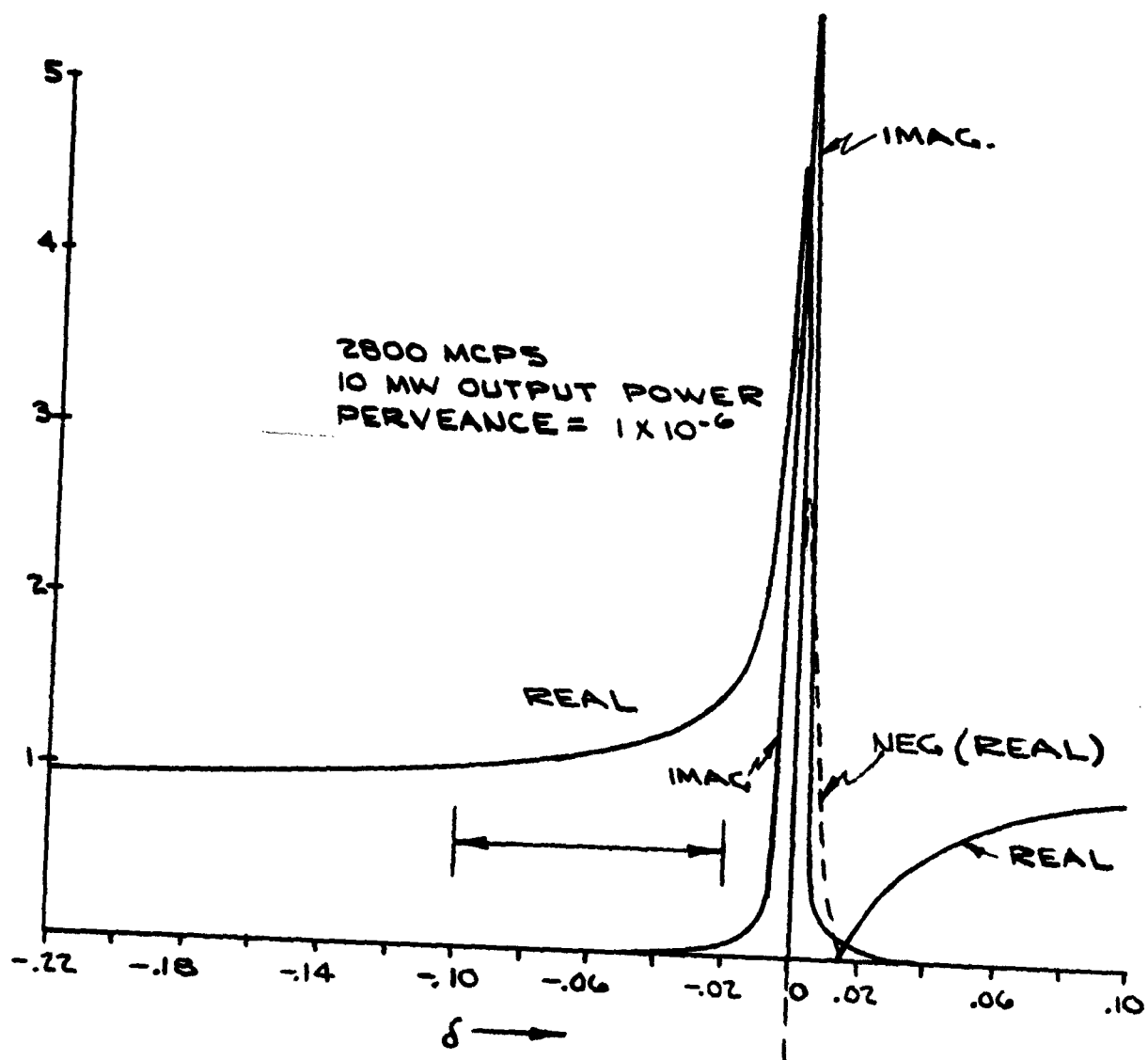Fig. 1: Real and imaginary parts of cosh u

17 RESONATORS

50

GAIN
db

40

12 RESONATORS

6% TUNING

30

7 RESONATORS

20

10

-.10

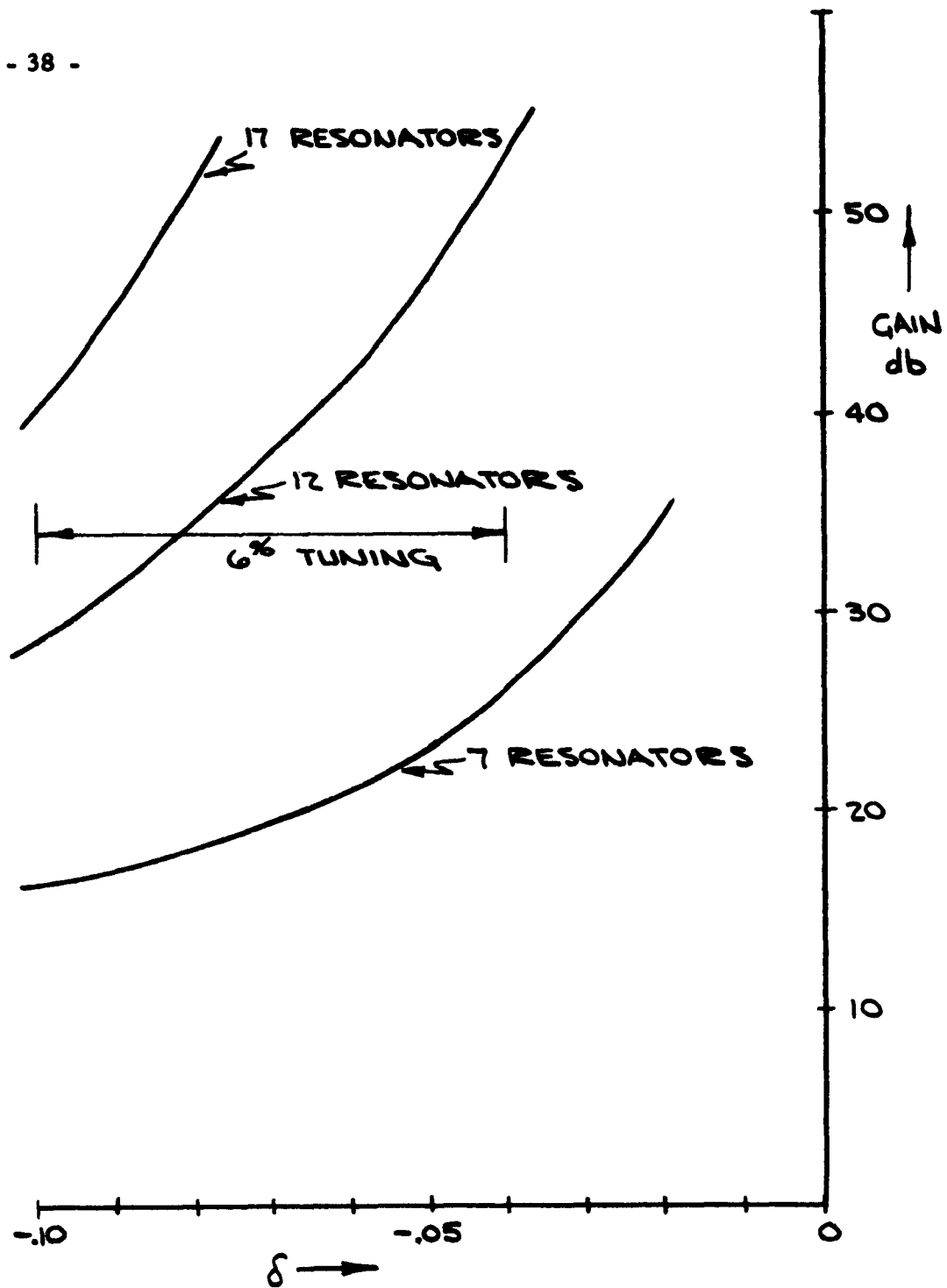-.05

0

$\delta \longrightarrow$

Fig. 2: Gain versus frequency
for multiresonator klystrons in
flat phase vs frequency region

and closely spaced. The drift length used in Fig. 2 is 2 inches so it can be seen that very useful tubes can be built in lengths of two feet. We can compare this with tubes which provide 45 db of gain in 5 feet of length. This shows the dramatic improvement available in terms of gain per unit length by the use of numerous closely spaced resonators. It can also be seen in Fig. 1 that there is a near zero of gain at a frequency of about 1-1/2% above the resonant frequency of each cavity. This null would be so located in stagger tuning as to radically reduce in size the gain peak which would be obtained in the case of degenerate tuning.

It is important to notice how slowly the real part of cosh u (in Fig. 1) sinks down through the value unity as detuning toward lower frequencies takes place. There is less than 10% change in going from $\delta = .1$ to $-.2$, for example. Now the magnitude of cosh u in this region is sensitive to the value of perveance used in the beam. We see that we should have used a beam microperveance of about 2. If so, we would have obtained a curve showing the real part of cosh u remaining appreciably above unity for more than 15% in frequency range. Thus we would have obtained gain magnitude curves such as those of Fig. 2 which are higher and much flatter in value. In addition we would have found ourselves with much more room for stagger tuning of the resonators. It is regretted that the pressure of time has prevented our carrying out such computation.

In a tube such as we are describing it is important that the many short drift tubes present no feedback possibilities due to drift tube propagation at harmonic or other frequencies. This is especially important in the vicinity of the resonant frequencies where gain may be high. Figure 3 shows how drift tube dimensions are chosen.
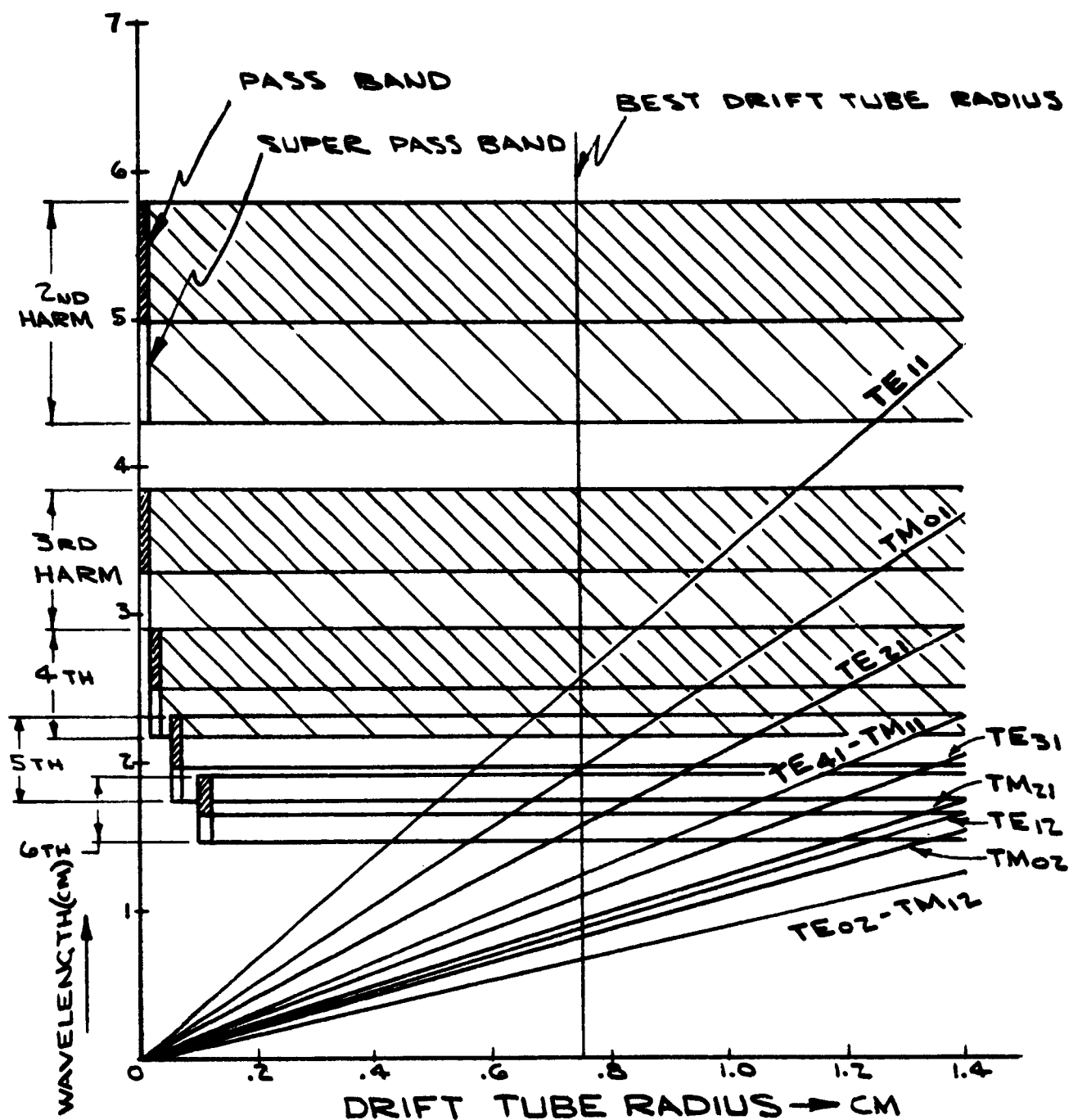
Fig. 3: Drift tube cut-off wavelengths and
harmonics of the operating frequencies

V. High Level and Output Stages

No trustworthy or tractable design theory exists in a form useful for designing the high level klystron stages. In our previous report[3] we treated these by reference to empirical evidence.

Also in our last report we developed expressions for a multi-element output section capable of presenting to the klystron beam an optimum value of output gap impedance over an adequate band of frequencies. We find no need to expand upon what has been presented. Some concern has been expressed to the author regarding the bandwidth predicted. This concern derived from comparison of predictions of bandwidth capability with similar predictions for duplexer devices which, of course, must be matched to waveguide impedances throughout the band. This is not the case in the output of a modern high-power klystron, however. We always desire a large VSWR seen looking back into such a klystron. Without this, tube efficiency would either be severely reduced in one case, or else the efficiency would be cut in half and excessive r-f voltage would appear in the output resonator in the other case. In this latter case, electrons would be

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3. RADC-TDR-62-199.

reflected back through the drift tube in a manner which would tend to cause spurious oscillations or other erratic tube behavior.

Some brief experimental work was done during this program toward obtaining measured curves of broadband output resonator performance. This work has not culminated in presentable data. This is due to a poor quality of r-f contacts used in the cold test assembly. It was considered to be not of sufficient interest to undertake the expense of preparing soldered output sections in order to demonstrate what is believed to be most soundly predicated on well-known information.

VI. Infinite Beam Multigap Theory

1. Introduction and Summary

We present here the results of some work on the modulation of a beam by many gaps. The goal was to obtain a large signal description of the multigap problem. The results obtained have been more modest due to difficulties in applying boundary conditions at the 2nd, 3rd, etc., gaps.

The infinite beam model is used to formulate the multigap problem. The displacement vector formalism,[4] which has been successful in the large signal description[5] of a beam modulated at an infinitesimal extended gap, provides us with an integral equation for the displacement vector in the multigap situation. The integral equation for N-gaps is as follows.

$$\sigma(x,\theta) = \int_{-\infty}^{x} dx' \; \frac{\sin \lambda(x-x')}{\lambda} \sum_{i=1}^{N} \frac{\alpha_i}{2} \, S\left[x' + \sigma(x', \theta - x + x') - x_i \right] \cdot$$

$$\cdot \sin\left(\theta - x + x' + \phi_i\right)$$

$$(107)$$

4. RADC-TDR-62-199, IV Eulerian Theory, Section A, p. 61.

5. Ibid, Section C, p. 91.

In Eq. (107), $\sigma = \beta_e \zeta$ is the dimensionless displacement vector, $x = \beta_e z$, $\beta_e = \omega/v_0$, $a_i = V_i/V_0$, $\theta = \omega t$, and ϕ_i is a phase angle associated with the ith gap. This equation provides a convenient starting point for numerical calculations. We have used infinitesimally extended gaps but this is not necessary. For gaps with finite width, the same form is valid with the δ-functions replaced by functions characterizing the gaps.

We have solved Eq. (107), for the case of two gaps, within the framework of second order perturbation theory. That is, we have kept terms through $\mathcal{O}(a_i a_j)$. The result is

$$\sigma(x, \theta) = 0 \quad , \quad x < 0$$

$$= \frac{\sin \lambda x}{\lambda} \left\{ \left(\frac{\alpha_1}{2}\right) \sin(\theta - x) - \frac{1}{2}\left(\frac{\alpha_1}{2}\right)^2 \sin^2(\theta - x) \right\},$$
$$0 < x < x_2$$

$$= \frac{\sin \lambda x}{\lambda} \left\{ \left(\frac{\alpha_1}{2}\right) \sin(\theta - x) - \frac{1}{2}\left(\frac{\alpha_1}{2}\right)^2 \sin^2(\theta - x) \right\}$$

$$+ \frac{\sin \lambda(x - x_2)}{\lambda} \left\{ \left(\frac{\alpha_2}{2}\right) \sin\left[\theta - (x - x_2) + \phi_2\right] \right.$$

$$\left. - \frac{1}{2}\left(\frac{\alpha_2}{2}\right)^2 \sin^2\left[\theta - (x - x_2) + \phi_2\right] \right\}$$

$$+ (\text{next page})$$

$$(108)$$

$$- \left\{ \cos \lambda x_2 \left(\frac{\alpha_1}{2} \right) \sin(\theta - x) \right\} \left\{ \frac{\sin \lambda(x - x_2)}{\lambda} \left(\frac{\alpha_2}{2} \right) \sin \left[\theta - (x - x_2) + \phi_2 \right] \right\}$$

$$+ \left\{ \frac{\sin \lambda x_2}{\lambda} \left(\frac{\alpha_1}{2} \right) \sin(\theta - x) \right\} \left\{ \cos \lambda(x - x_2) \left(\frac{\alpha_2}{2} \right) \sin \left[\theta - (x - x_2) + \phi_2 \right] \right\}$$

$$- \left\{ \frac{\sin \lambda x_2}{\lambda} \left(\frac{\alpha_1}{2} \right) \sin(\theta - x) \right\} \left\{ \frac{\sin \lambda(x - x_2)}{\lambda} \left(\frac{\alpha_2}{2} \right) \cos \left[\theta - (x - x_2) + \phi_2 \right] \right\},$$

$$x > x_2$$

In Eq. (108) we have positioned the first gap at $x_1 = 0$ and assigned $\phi_1 = 0$.

This solution displays several interesting properties. The solution in the region $0 < x < x_2$ is just that obtained by expanding the large signal solution[6] through $\mathcal{O}(a_1^2)$. In the region $x > x_2$ the first two terms are just the super-position of the solutions which would result if we modulated the beam separately by each gap. The last three terms are interference terms and occur because modulation exists on the beam when it enters the second gap. The first two of these terms are what we would expect if at $x = x_2$ the beam had displacement and velocity modulation respectively in the amounts shown below.

$$\sigma \Big]_{x_2} = \frac{\sin \lambda x_2}{\lambda} \left(\frac{\alpha_1}{2} \right) \sin(\theta - x_2)$$

$$\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial \theta} \Big]_{x_2} = -\cos \lambda x_2 \left(\frac{\alpha_1}{2} \right) \sin(\theta - x_2)$$

$$(109)$$

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6. RADC-TDR-62-199, IV Eulerian Theory, Section C, Eq. (IVC-73), p. 126.

The last term is a contribution which seems to mix the arriving displacement with a second gap displacement whose electronic phase is 90° out of phase with the exciting voltage. The complexity of the solution presented by Eq. (108) serves to demonstrate the difficulties which arise in a large signal treatment of the multigap problem. It is the interference terms which cause the trouble. However, it is just these terms which are of great interest in the large signal situation.

The space-charge field, velocity field, current and charge density in the modulated beam can be calculated with the aid of formulas which have been previously derived.[7]

In part 2 we derive the integral equation (107) for the multigap problem.

In part 3 we calculate the second order perturbation theory solution given by Eq. (108).

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7. RADC-TDR-62-199, IV Eulerian Theory, Section C, part 5, pp. 128-131.

2. Integral Equation for the Multigap Problem

For a beam modulated at N infinitesimally extended gaps, the differential equation (8) for the displacement vector is

$$\left\{ \left[\partial/\partial\theta + \partial/\partial x \right]^2 + \lambda^2 \right\} \sigma$$

$$= \sum_{i=1}^{N} \frac{\alpha_i}{2} \, \delta \left[x + \sigma(x,\theta) - x_i \right] \sin(\theta + \phi_i) \tag{110}$$

where x_i, ϕ_i are the position and phase associated with the ith gap. We define

$$F(\theta,x) = \sum_{i=1}^{N} \frac{\alpha_i}{2} \, \delta \left[x + \sigma(x,\theta) - x_i \right] \sin(\theta + \phi_i) \tag{111}$$

Express $\sigma(x, \theta)$ in the form

$$\sigma(x,\theta) = \sum_{r} \sigma_r(x) \, e^{jr(\theta - x)} \tag{112}$$

Eq. (110) then yields

$$\frac{d^2\sigma_r}{dx^2} + \lambda^2 \sigma_r = F_r(x) \tag{113}$$

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8. RADC-TDR-62-199, IV Eulerian Theory, Section C, Eq. (IVC-56).

$$F_r(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-jr(\theta - x)} F(\theta, x) \, d\theta \tag{114}$$

We solve Eq. (113) by Green's function methods. That is, consider the function G,

$$\left(\frac{d^2}{dx^2} + \lambda^2 \right) G(x, x') = \delta(x - x') \tag{115}$$

$$\sigma_r(x) = \int_{-\infty}^{+\infty} G(x, x') F_r(x') \, dx' \tag{116}$$

Thus G is a solution to

$$\left(\frac{d^2}{dx^2} + \lambda^2 \right) G(x, x') = 0 \tag{117}$$

except at the point $x = x'$. The solution for G which satisfies the right boundary conditions is

$$G(x - x') = \left\{ \begin{array}{ll} 0, & x - x' < 0 \\ \dfrac{\sin \lambda (x - x')}{\lambda}, & x - x' > 0 \end{array} \right\} \tag{118}$$

The boundary conditions have been chosen so that the solution $\sigma_r(x)$., .q. (116) is not affected by the ith gap unless

$x > x_i$. With Eqs. (118), (116), and (112) we obtain

$$\sigma(x,\theta) = \int_{-\infty}^{x} \frac{\sin \lambda(x-x')}{\lambda} \sum_{\gamma} F_{\gamma}(x') \, e^{j\gamma(\theta-x)} dx' \qquad (119)$$

However, from Eq. (114)

$$F(x,\theta) = \sum_{\gamma} e^{j\gamma(\theta-x)} F_{\gamma}(x) \qquad (120)$$

and therefore

$$\sum_{\gamma} F_{\gamma}(x') \, e^{j\gamma\{[\theta-x+x']-x'\}} = F(x', \theta-x+x') \qquad (121)$$

Hence, Eq. (119) takes the form

$$\sigma(x,\theta) = \int_{-\infty}^{x} \frac{\sin \lambda(x-x')}{\lambda} F(x', \theta-x+x') \, dx' \qquad (122)$$

Substituting the expression (111) for F in Eq. (122) we obtain the desired integral equation (107).

3. Second Order Perturbation Theory for the Two Gap Problem

We expand σ in power series in the a_i.

$$\sigma(x,\theta) = \sum_{\ell,h=1}^{\infty} \alpha_1^{\ell} \alpha_2^{h} \, \sigma_{\ell+h}(x,\theta) \tag{123}$$

Through second order Eq. (107) becomes

$$\sigma(x,\theta) = \int_{-\infty}^{x} dx' \frac{\sin\lambda(x-x')}{\lambda} \left\{ \left[\frac{\alpha_1}{2} \delta(x') + \frac{\alpha_1}{2} \delta'(x') \sigma(x';\theta-x+x') \right] \cdot \right.$$

$$\cdot \sin(\theta-x+x') \tag{124}$$

$$+ \left[\frac{\alpha_2}{2} \delta(x-x_2) + \frac{\alpha_2}{2} \delta'(x-x_2) \sigma(x',\theta-x+x') \right] \cdot$$

$$\left. \cdot \sin(\theta-x+x'+\phi_2) \right\}$$

Integrating the δ' terms by parts and calculating the first

order contribution we get

$$\sigma = 0 \quad , \quad x < 0$$

$$\sigma = \frac{\alpha_1}{2} \frac{\sin\lambda x}{\lambda} \sin(\theta-x) , \quad 0 < x < x_2$$

$$\tag{125}$$

$$\sigma = \frac{\alpha_1}{2} \frac{\sin\lambda x}{\lambda} \sin(\theta-x) + \frac{\alpha_2}{2} \frac{\sin\lambda(x-x_2)}{\lambda} \sin(\theta-x+x_2+\phi_2) ,$$

$$x > x_2$$

We now substitute this first order solution into the right-hand side of Eq. (124) and perform the indicated integrations. The result is given by Eq. (108). The factor 1/2 which multiplies the quantities

$$\left(\alpha_{1/2}\right)^2 \frac{\sin \lambda x}{\lambda} \sin^2 (\theta - x)$$

$$\left(\frac{\alpha_2}{2}\right)^2 \frac{\sin \lambda (x - x_2)}{\lambda} \sin^2 \left[\theta - (x - x_2) + \phi_2\right] \tag{126}$$

arises because the derivative

$$\frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_1}{\partial \theta} \tag{127}$$

jumps from zero to $(a_1/2) \sin \theta$ at $x = 0$ and the δ-function evaluates this quantity at its average value, $(a_1/4) \sin \theta$.

VII. Large-Signal Disc Model Theory

1. Introduction and Summary

We consider the disc model from the large-signal point of view. For the infinite beam model the space-charge forces are characterized by the Green's function

$$K(x-x') = \frac{1}{2}|x-x'| \qquad (128)$$

while for the disc model we have

$$\left.\begin{array}{l} K(x-x') = \dfrac{1}{2\gamma}\, e^{-\gamma|x-x'|} \\[2mm] \gamma = \dfrac{2}{\beta_e b} \\[2mm] \beta_e = \dfrac{\omega}{\nu_0} \\[2mm] b = \text{beam radius} \end{array}\right\} \qquad (129)$$

We derive in part 2 an equation of motion for the disc model which is valid when the signal level is large. The result is

$$\left\{\left[\partial_\theta + \partial_x\right]^2 + \lambda^2\right\}\sigma - \lambda^2\int_{-\infty}^{+\infty} dx'\,\frac{\gamma}{2}\, e^{-\gamma|x-x'|}\cdot \sigma(x',\theta)$$

$$= -\alpha\, F_q(x+\sigma,\theta) - \lambda^2\sum_{k=1}^{\infty}\mathcal{R}_k \qquad (130)$$

where

$$\mathcal{R}_k = \frac{1}{(k+1)!} \int_{-\infty}^{+\infty} dx' \frac{y}{2} e^{-y|x-x'|} \left\{ (\tfrac{1}{2}x')^k (\sigma - \sigma')^{k+1} \right\}$$

(131)

is a remainder term which vanishes in the case of the infinite

beam model. It is conjectured that the remainder terms are

small and that a good approximation for the disc model may

be obtained by neglecting these terms in Eq. (130).

In part 3 we investigate the dispersive properties of the

waves characterized by the disc model equation of motion. We

find that, while the infinite beam model gives rise to two non-

dispersive space-charge waves, the disc model on the other

hand leads to four waves, all of which are dispersive. The

wave numbers in dimensionless units are as follows.

$$\beta_{1,2} = \omega \left\{ 1 \pm \frac{\lambda}{(\gamma^2 + \omega^2)^{1/2}} \right\}$$

$$\beta_{3,4} = \pm j \gamma \left\{ 1 + \frac{\lambda^2}{2} \frac{(\omega^2 - \gamma^2)}{(\omega^2 + \gamma^2)^2} \right\} - \frac{\lambda^2 \omega \gamma^2}{(\omega^2 + \gamma^2)^2}$$

(132)

Solutions 1, 2 are space-charge waves and solutions 3, 4 are

evanescent waves which damp exponentially.

The dispersive properties of the disc model space-

charge waves have the effect of providing a different space-

charge reduction factor for each harmonic, r.

$$\lambda = \frac{\omega_p}{\omega} \longrightarrow \lambda_{gr} = \frac{R_r \omega_p}{\omega}$$

$$R_r = \frac{r}{(r^2 + y^2)^{1/2}}$$

$$\left.\begin{array}{c}\\[2em]\end{array}\right\} \quad (133)$$

This has the effect of distorting the form of the disturbance on the beam as it travels down the drift space.

2. Large-Signal Equation of Motion
   for the Disc Model

For one-dimensional flow[9] the equation of motion is

$$\left[\partial/\partial x + \partial/\partial \theta\right]^2 \sigma(x, \theta) = -\alpha F_g(x+\sigma, \theta)$$

$$-\lambda^2 \int \frac{\partial K(x, x')}{\partial x}\bigg] dx' \qquad (134)$$

$$x \to x+\sigma$$
$$x' \to x'+\sigma'$$

In the following we treat $K(x, x')$ as a function defined as the limit of a sequence of well-behaved functions. These functions are infinitely differentiable.

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9. RADC-TDR-62-199, IV Eulerian Theory, Section C,
p. 93.

Consider the term

$$I = \lambda^2 \int \left. \frac{\partial k(x,x')}{\partial x} \right]_{\substack{x \to x+\sigma \\ x' \to x'+\sigma'}} dx' \qquad (135)$$

The Taylor operators allow us to write Eq. (135) in the form

$$I = \lambda^2 \int e^{\sigma \cdot \partial/\partial x} \, e^{\sigma' \cdot \partial/\partial x'} \, \frac{\partial k(x,x')}{\partial x} \, dx' \qquad (136)$$

Expanding $e^{\sigma' \partial/\partial x'}$ and integrating the nth term n-times by parts we get

$$I = \lambda^2 \int e^{\sigma \cdot \partial/\partial x} \, \frac{\partial k(x,x')}{\partial x} \, e^{-\partial/\partial x' \cdot \sigma'} \, dx' \qquad (137)$$

where

$$e^{-\partial/\partial x' \cdot \sigma'} \equiv 1 - \frac{\partial}{\partial x'}(\sigma') + \frac{1}{2!}\frac{\partial^2}{\partial x'^2}(\sigma')^2 + \cdots \qquad (138)$$

Eq. (137) can be written

$$I = \lambda^2 \int e^{\sigma \cdot \partial/\partial x} \, \frac{\partial k(x,x')}{\partial x} \left(e^{-\partial/\partial x' \cdot \sigma'} - 1 \right) dx' \qquad (139)$$

because

$$\int \frac{\partial k(x,x')}{\partial x} \, dx' = -\int \frac{\partial k(x,x')}{\partial x'} \, dx' = 0 \tag{140}$$

This follows from the property

$$k(x, x') = k(x - x') \xrightarrow[x' \to \pm\infty]{} 0 \tag{141}$$

We expand the exponentials in Eq. (139)

$$I = \lambda^2 \int \sum_{n=0}^{\infty} \frac{\sigma^n (\frac{\partial}{\partial x})^n}{n!} \cdot \frac{\partial k(x,x')}{\partial x} \left(\sum_{m=1}^{\infty} \frac{(-)^m (\frac{\partial}{\partial x'})^m (\sigma')^m}{m!} \right) dx' \tag{142}$$

The operation $(\partial/\partial x)^n$ on K is the same as $(-\partial/\partial x')^n$ on K. Integrating by parts, n-1 times, Eq. (142) becomes

$$I = \lambda^2 \int \sum_{n=0}^{\infty} \frac{(\sigma)^n}{n!} \cdot \frac{\partial^2 k(x,x')}{\partial x^2} \left(\sum_{m=1}^{\infty} \frac{(-)^m (\frac{\partial}{\partial x'})^{n+m-1} (\sigma')^m}{m!} \right) dx' \tag{143}$$

Introduce a new summation index

$$
\left.\begin{array}{l}
k = n + m - 1 \\[4pt]
1 \le m \le \infty \\[4pt]
m - 1 \le k \le \infty
\end{array}\right\} \tag{144}
$$

$$
I = \lambda^2 \int dx' \cdot \frac{\partial^2 k(x,x')}{\partial x^2} \sum_{m=1}^{\infty} \sum_{k=m-1}^{\infty} \frac{(-)^m \, \sigma^{k-m+1} \left(\tfrac{\partial}{\partial} x'\right)^k (\sigma')^m}{(k-m+1)! \, m!}
$$

We now interchange the order of the sums
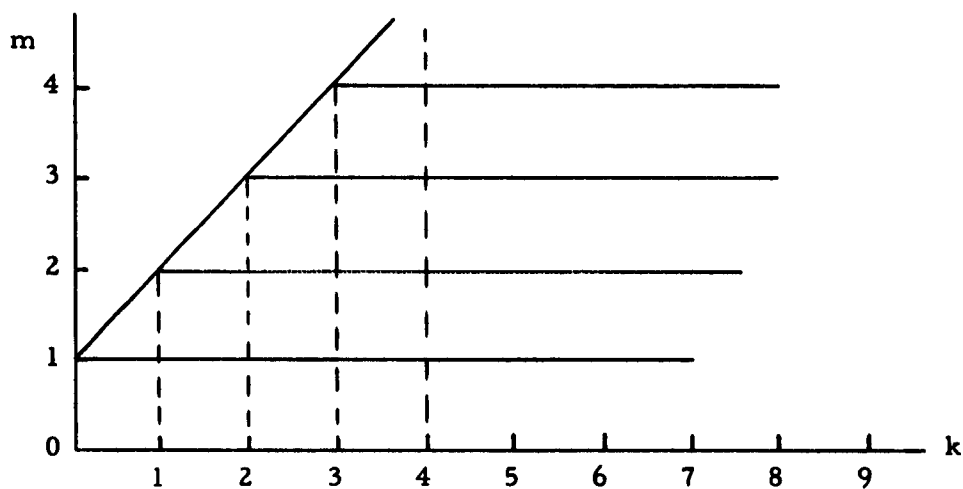
$$
\left.\begin{array}{l}
0 < k < \infty \\[4pt]
1 < m < k + 1
\end{array}\right\} \tag{145}
$$

Finally we get

$$I = \lambda^2 \int dx' \frac{\partial^2 K(x,x')}{\partial x^2} \sum_{k=0}^{\infty} \sum_{m=1}^{k+1} \frac{(-)^m \sigma^{k-m+1} \left(\frac{\partial}{\partial x}\right)^k (\sigma')^m}{(k-m+1)! \, m!} \qquad (146)$$

In the case of the infinite beam model

$$\frac{\partial^2 K(x,x')}{\partial x^2} = - \delta(x-x') \qquad (147)$$

and Eq. (146) becomes

$$I = - \lambda^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{m=1}^{k+1} \binom{k+1}{m} (-)^m \sigma^{k-m+1} \left(\frac{\partial}{\partial x}\right)^k (\sigma)^m$$

$$(148)$$

$$= \lambda^2 \sigma - \lambda^2 \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \sum_{m=1}^{k+1} \binom{k+1}{m} (-)^m \sigma^{k-m+1} \left(\frac{\partial}{\partial x}\right)^k (\sigma)^m$$

We show that the second term in Eq. (148) vanishes.

To demonstrate this we show that

$$\sum_{m=1}^{k+1} \binom{k+1}{m} (-)^m \sigma^{k-m+1} \left(\frac{\partial}{\partial x}\right)^k (\sigma)^m = 0 \, ; \, k = 1, 2, \cdots \qquad (149)$$

We can include the term $m = 0$ since $\partial/\partial x(1) = 0$, and hence
we wish to prove the following lemma.

$$G(k, k-1) = \sum_{m=0}^{k} \binom{k}{m}(-)^m \sigma^{k-m} \left(\frac{\partial}{\partial x}\right)^{k-1} (\sigma)^m = 0 \quad (150)$$

In general, we define $G(k, \ell)$ by

$$G(k, \ell) = \sum_{m=0}^{k} \binom{k}{m}(-)^m \sigma^{k-m} \left(\frac{\partial}{\partial x}\right)^{\ell} \sigma^m . \quad (151)$$

First we develop a recursion relation

$$\frac{\partial G(k, \ell)}{\partial x} = \sum_{m=0}^{k} \binom{k}{m}(-)^m \left\{ \frac{\partial \sigma^{k-m}}{\partial x} \frac{\partial^\ell \sigma^m}{\partial x^\ell} + \sigma^{k-m} \frac{\partial^{\ell+1} \sigma^m}{\partial x^{\ell+1}} \right\}$$

$$= \sum_{m=0}^{k-1} \frac{k!(k-m)}{(k-m)! m!}(-)^m \frac{\partial \sigma}{\partial x} \sigma^{k-m-1} \left(\frac{\partial}{\partial x}\right)^{\ell} \sigma^m + G(k, \ell+1) \quad (152)$$

$$= k \frac{\partial \sigma}{\partial x} G(k-1, \ell) + G(k, \ell+1)$$

Now consider

$$G(k, 0) = (\sigma - \sigma)^k = 0 \quad , \quad k = 1, 2, \cdots \quad (153)$$

and apply the recursion relation.

$$0 = \frac{\partial G(k,0)}{\partial x} = h \frac{\partial \sigma}{\partial x} G(k-1,0) + G(k,1) \qquad (154)$$

However

$$G(k-1,0) = 0 \; ; \; k = 2, 3, \cdots \qquad (155)$$

so

$$G(k,1) = 0 \; ; \; k = 2, 3, \cdots \qquad (156)$$

Apply the same reasoning to G(k, 1). We have

$$0 = \frac{\partial G(k,1)}{\partial x} = h \frac{\partial \sigma}{\partial x} G(k-1,1) + G(k,2) \qquad (157)$$

but

$$G(k-1,1) = 0 \; ; \; k = 3, 4, \cdots \qquad (158)$$

and hence

$$G(k,2) = 0 \; ; \; k = 3, 4, \cdots \qquad (159)$$

By induction we obtain

$$\therefore \quad G(k, \ell) = 0 \; ; \; k = \ell+1, \ell+2, \cdots \qquad (160)$$

and a special case of this relation is the desired lemma,

$$G(k, k-1) = 0 \qquad (161)$$

which lies, so to speak, on the edge of the hierarchy (160).

We have proved that Eq. (148) reduces to

$$I = \lambda^2 \sigma \tag{162}$$

Thus for the infinite beam model Eq. (134) becomes

$$\left\{ \left[\partial_{\theta} + \partial_x \right]^2 + \lambda^2 \right\} \sigma = -\alpha F_q (x + \sigma, \theta) \tag{163}$$

This equation was derived in a more physical way in a previous report.[10] The development above is included here because it is needed for the disc-model discussion below.

For the disc-model we return to Eq. (146) which applies to any one-dimensional model. The disc-model Green's function[11] is

$$k(x-x') = \left(\frac{1}{2}\right)\left(\frac{\beta_e b}{2}\right) e^{-\left(\frac{2}{\beta_e b}\right)|x-x'|} \tag{164}$$

$$\frac{\partial^2 k(x-x')}{\partial x^2} = -\delta(x-x') + \frac{1}{(\beta_e b)} e^{-\left(\frac{2}{\beta_e b}\right)|x-x'|} \tag{165}$$

10. RADC-TDR-62-199, IV Eulerian Theory, Section C, pp. 112-121.

11. R. N. Wilson, "Large Signal Space-Charge Theory of Klystron Bunching," M. L. Report No. 750, Microwave Lab., Stanford University, Calif., September 1960, p. 109.

We substitute Eq. (165) into Eq. (146). The result is

$$(I)_{disc.} = (I)_{inf.\,beam} + \lambda^2 \int \frac{1}{(\rho_e b)} e^{-\left(\frac{2}{\rho_e b}\right)|x-x'|} \sum_{k=0}^{\infty} \sum_{m=1}^{k+1} \frac{(-)^m \sigma^{k-m+1}(\frac{1}{3}x')(\sigma')^m}{(k-m+1)!\, m!} \quad (166)$$

The last term in Eq. (166) can be rewritten as follows:

$$\lambda^2 \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \int dx' \frac{1}{(\rho_e b)} e^{-\left(\frac{2}{\rho_e b}\right)|x-x'|} (\tfrac{1}{3}x')^k (\sigma - \sigma')^{k+1}$$

$$\hspace{6cm} (167)$$

$$- \lambda^2 \int \frac{dx'}{(\rho_e b)} e^{-\left(\frac{2}{\rho_e b}\right)|x-x'|} \cdot \sigma(x',\theta)$$

Combining Eq. (167) with Eq. (166) the disc-model equation of motion becomes

$$\left\{ \left[\tfrac{1}{3\theta} + \tfrac{1}{3x}\right]^2 + \lambda^2 \right\} \sigma - \lambda^2 \int \frac{dx'}{(\rho_e b)} e^{-\left(\frac{2}{\rho_e b}\right)|x-x'|} \cdot \sigma(x',\theta)$$

$$\hspace{6cm} (168)$$

$$= -\alpha F_3(x+\sigma,\theta) - \lambda^2 \sum_{k=1}^{\infty} \mathcal{R}_k$$

where

$$\mathcal{R}_k = \frac{1}{(k+1)!} \int_{-\infty}^{+\infty} dx' \frac{1}{(\rho_e b)} e^{-\left(\frac{2}{\rho_e b}\right)|x-x'|} \cdot \left\{ (\tfrac{1}{3}x')^k (\sigma-\sigma')^{k+1} \right\} \quad (169)$$

We note that the integrand of Eq. (169) vanishes at $x = x'$, but the peak of the weighting function, W, lies at $x = x'$.

$$W = \frac{1}{(\beta e b)} \; e^{-\left(\frac{2}{\beta e b}\right)|x-x'|}$$

(170)

$$\int_{-\infty}^{+\infty} W(x,x')\, dx' = 1$$

(171)

The quantity $\mathcal{R}_k$ is a weighted average of the quantity

$$\frac{\left(\frac{1}{3}x'\right)^k (\sigma-\sigma')^{k+1}}{(k+1)!}$$

(172)

about the value $x' = x$, with weighting function W. $\mathcal{R}_k$ vanishes in the two limits

$$\left. \begin{array}{l} b \longrightarrow 0, \quad W \longrightarrow \delta(x-x') \\[2mm] b \longrightarrow \infty, \quad W \longrightarrow 0 \end{array} \right\}$$

(173)

where b is the beam diameter. We conjecture that

$$\lambda^2 \sum_{k=1}^{\infty} \mathcal{R}_k$$

(174)

is small compared to the other terms in Eq. (168). In this case a good approximation to the disc-model equation of

motion is

$$\left\{ \left[\partial/\partial\theta + \partial/\partial x \right]^2 + \lambda^2 \right\} \sigma - \lambda^2 \int_{-\infty}^{+\infty} \frac{dx'}{(\beta_e b)} \; e^{-\left(\frac{2}{\beta_e b}\right)|x-x'|} \cdot \sigma(x'; \theta)$$

(175)

$$= -\alpha \, F_3 \, (x+\sigma, \, \theta)$$

3. Dispersion Characteristics of the Approximate Large-Signal Disc Model

To determine the type of waves which the disc-model supports, we deal with the homogeneous version of Eq. (175) and express $\sigma(x, \theta)$ in the form

$$\sigma(x, \theta) = \int d\beta \, d\omega \; e^{j(\omega\theta - \beta x)} \, \sigma(\omega, \beta)$$

(176)

Also we express the kernel function in the integral term of Eq. (175) in terms of its Fourier transform.

$$\frac{1}{(\beta_e b)} \, e^{-\left(\frac{2}{\beta_e b}\right)|x-x'|} = \frac{2}{\pi} \left(\frac{1}{\beta_e b}\right)^2 \int_{-\infty}^{+\infty} \frac{e^{-j\beta(x-x')} d\beta}{\left(\frac{2}{\beta_e b}\right)^2 + \beta^2}$$

(177)

Upon substitution into the homogeneous form of Eq. (175) the dispersion relation which results is

$$(\omega - \beta)^2 (\gamma^2 + \rho^2) - \lambda^2 \rho^2 = 0$$

$$\gamma = \frac{\lambda}{\rho_e b}$$

$$\left. \vphantom{\begin{matrix}a\\b\end{matrix}} \right\} \tag{178}$$

If we solve for $\beta(\omega)$ we obtain four roots. Through leading order in λ these are

$$\beta_{1,2} = \omega \left\{ 1 \pm \frac{\lambda}{(\gamma^2 + \omega^2)^{1/2}} \right\}$$

$$\beta_{3,4} = \pm j\gamma \left\{ 1 + \frac{\lambda^2}{2} \frac{(\omega^2 - \gamma^2)}{(\omega^2 + \gamma^2)^2} \right\} - \frac{\lambda^2 \omega \gamma^2}{(\omega^2 + \gamma^2)^2} \tag{179}$$

The waves $\beta_{1,2}$ are space-charge waves and $\beta_{3,4}$ are evanescent waves.[12] Thus whereas there were only two non-dispersive space-charge waves for the infinite beam model, we now have, in addition, two evanescent waves. Also all four of the waves characterized by Eq. (179) are dispersive and this makes the situation more complicated.

For modulation by an infinitesimally extended gap a first order solution has previously been obtained.[13] We shall here neglect the evanescent waves and match boundary

12. R. N. Wilson, M. L. Report No. 750, pp. 112-116.
13. Ibid, pp. 112-113.

conditions in the same way as for the infinite beam model.[14] We let

$$
\left.
\begin{array}{l}
\sigma(0,\theta) = 0 \\[2mm]
\left.\dfrac{\partial\sigma}{\partial x} + \dfrac{\partial\sigma}{\partial\theta}\right]_{x=0} = \left[(1+\alpha\sin\theta)^{\frac{1}{4}} - 1\right]
\end{array}
\right\}
\tag{180}
$$

and approximate the second of Eqs. (180) by

$$
\left.\dfrac{\partial\sigma}{\partial x} + \dfrac{\partial\sigma}{\partial\theta}\right]_{x=0} \simeq (\alpha/4)\sin\theta
\tag{181}
$$

We obtain the following solution.

$$
\sigma(x,\theta) =
\begin{cases}
0, & x < 0 \\[2mm]
\dfrac{\alpha}{2}\left[\dfrac{\sin\frac{\lambda x}{(1+\gamma^2)^{1/2}}}{\frac{\lambda}{(1+\gamma^2)^{1/2}}}\right]\sin(\theta-x), & x > 0
\end{cases}
\tag{182}
$$

$$
\gamma = \frac{2}{\beta_e b}
$$

This solution agrees with the solution obtained before, at points down the drift space such that the evanescent waves (exponentially damped) are negligible.

This suggests that a good approximate solution is provided by neglecting the evanescent waves when we fit

14. RADC-TDR-62-199, IV Eulerian Theory, Section C, p. 124.

the boundary conditions. The resulting solution is meaning-
ful only at distances from the gap which are large enough so
the evanescent waves can be neglected.

We approximate Eq. (180) by

$$\sigma(0,\theta) = 0$$

$$\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial \theta}\Bigg]_{x=0} = \frac{\alpha}{2}\sin\theta - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2\sin^2\theta \qquad (183)$$

$$= \frac{\alpha}{2}\sin\theta - \frac{1}{4}\left(\frac{\alpha}{2}\right)^2\left[1 - \cos 2\theta\right]$$

The d-c part of Eq. (183) gives rise only to evanescent waves.
The solution is

$$\sigma(x,\theta) = \begin{cases} 0 \quad, \; x < 0 \\[2mm] \left(\frac{\alpha}{2}\right)\left[\dfrac{\sin\frac{\lambda x}{(1+\gamma^2)^{1/2}}}{\dfrac{\lambda}{(1+\gamma^2)^{1/2}}}\right]\sin(\theta-x) + \frac{1}{4}\left(\frac{\alpha}{2}\right)^2\left[\dfrac{\sin\frac{2\lambda x}{(4+\gamma^2)^{1/2}}}{\dfrac{2\lambda}{(4+\gamma^2)^{1/2}}}\right] \cdot \quad (184) \\[6mm] \qquad\qquad\qquad\qquad\qquad \cdot \cos 2(\theta-x), \; x > 0 \end{cases}$$

Eq. (184) shows that there is a different space-charge reduction
factor for each harmonic.

$$\left.\begin{aligned} \lambda_{gr} &= \frac{\omega_{gr}}{\omega} = \frac{R_r\,\omega_p}{\omega} \\[4mm] R_r &= \frac{r}{(r^2+\gamma^2)^{1/2}} \\[4mm] \gamma &= \frac{\lambda}{\beta_e b} \end{aligned}\right\} \qquad (185)$$

In the case of the infinite beam, $R_r = 1$, we found that the disturbance on the beam, caused by the modulation, traveled down the drift space without change in form but changed in amplitude by the multiplicative factor

$$\frac{\sin \lambda x}{\lambda} \tag{186}$$

This is no longer true with the disc model as shown by Eq. (184). Each harmonic has a different space-charge multiplicative factor

$$\frac{\sin \frac{r \lambda x}{(r^2+s^2)^{1/2}}}{\frac{s \lambda}{(r^2+s^2)^{1/2}}} \tag{187}$$

and the reason for this behavior lies in the dispersive properties of the space-charge waves.

VIII. Statistical Mechanical Formulation
of the Infinite Beam Model

1. Introduction and Summary

We formulate the infinite beam problem within the frame-work of statistical mechanics. We start with the statement of Liouville's theorem for the many particle density function and derive from this an equation of motion for the electron beam. The result is

$$\left[\frac{\partial}{\partial t} + (v_0 + \overline{\Delta v}) \frac{\partial}{\partial t} \right] \overline{\Delta v} \; - \; \frac{e}{m} \, E(z,t)$$

$$= \; \frac{1}{(n+\delta n)} \, \frac{\partial}{\partial t} \left\{ (n+\delta n) \left[(\overline{\Delta v})^2 - \overline{(\Delta v^2)} \right] \right\} \qquad (188)$$

$$\left. \begin{array}{c} \overline{\Delta v} \; = \; \overline{v - v_0} \\[2mm] \overline{\Delta v^2} \; = \; \overline{(v - v_0)^2} \end{array} \right\} \qquad\qquad (189)$$

$$v_0 \; = \; \text{d-c beam velocity}$$
$$e(n + \delta n) \; = \; \text{electron charge density} \qquad (190)$$

The term on the right is a diffusion term which appears to become large at and near crossover.

We conclude that the infinite beam model, which pictures the beam as a set of charge planes, ceases to be valid near crossover.

2. Derivation of the Hydrodynamical Equations of Motion

We start with the many particle distribution function,[15]

$$W(\vec{q}_1, \cdots, \vec{q}_n, \vec{Q}_1, \cdots, \vec{Q}_n, \vec{p}_1, \cdots, \vec{p}_n, \vec{P}_1, \cdots \vec{P}_n, t) \qquad (191)$$

for two species of particles, electrons (q_i, p_i) and ions (Q_i, P_i). According to Liouville's theorem the total time derivative of W vanishes.

$$\frac{\partial W}{\partial t} + \sum_i \frac{\partial W}{\partial \vec{q}_i} \dot{\vec{q}}_i + \sum_i \frac{\partial W}{\partial \vec{p}_i} \dot{\vec{p}}_i + \sum_i \frac{\partial W}{\partial \vec{Q}_i} \dot{\vec{Q}}_i + \sum_i \frac{\partial W}{\partial \vec{P}_i} \dot{\vec{P}}_i = 0 \qquad (192)$$

The positions q_i, Q_i and momenta p_i, P_i satisfy

$$\left. \begin{array}{ll} \dot{\vec{q}}_i = \vec{p}_i / m & , \quad \dot{\vec{p}}_i = e\left\{\vec{E} + \frac{\vec{p}_i}{m} \times \vec{B}\right\} \\[3mm] \dot{\vec{Q}}_i = \vec{P}_i / M & , \quad \dot{\vec{P}}_i = -e\left\{\vec{E} + \frac{\vec{P}_i}{M} \times \vec{B}\right\} \end{array} \right\} \qquad (193)$$

15. R. C. Tolman, The Principles of Statistical Mechanics, Oxford Univ. Press, New York, N.Y., 1st edition, pp. 48-52.

We define the single particle electron distribution function by

$$w(q_1, p_1, t) = \int W(q_i, Q_i, p_i, P_i)\, d^3q_2 \cdots d^3q_n\, d^3p_2 \cdots d^3p_n\, d^3q_i\, d^3p_i \quad (194)$$

and normalize the density function W so that

$$\int w\, d^3p = n(x, t) \quad , \qquad \left.\begin{array}{l} \vec{x} \equiv \vec{q}_1 \\ \vec{p} \equiv \vec{p}_1 \end{array}\right\} \qquad (195)$$

$$\int n(\vec{x}, t)\, d^3x = n \qquad (196)$$

Eq. (195) gives the electron particle density and Eq. (196) gives the total number of electrons.

Integrate Eq. (192) over all coordinates except q_1, p_1. The result is

$$\frac{\partial w(\vec{x}, \vec{p}, t)}{\partial t} + \sum_{i=1}^{3} \frac{\partial w}{\partial x_i} \frac{p_i}{m} + \sum_{i=1}^{3} \frac{\partial w}{\partial p_i}\, \dot{p}_i = 0 \qquad (197)$$

All other terms vanish since, for example

$$\int \sum_{i=1}^{3} \frac{\partial W}{\partial q_i}\, \dot{q}_i\, d^3q_2 \cdots d^3P_n = \sum_{i=1}^{3} \frac{\partial w(\vec{x}, \vec{p})}{\partial x_i} \frac{p_i}{m}$$

$$\qquad (198)$$

$$+ \sum_{i=4}^{3n} \int \frac{\partial W}{\partial q_i} \frac{p_i}{m}\, d^3q_2 \cdots d^3P_n$$

However, the last term vanishes because W is assumed
to vanish at the boundary of integration in the phase space.

We substitute for p_i in Eq. (197), from Eq. (193) and
obtain

$$\frac{\partial \omega}{\partial t} + \sum_{i=1}^{3} \frac{\partial \omega}{\partial x_i} \frac{p_i}{m} + \sum_{i=1}^{3} \frac{\partial \omega}{\partial p_i} e \left\{ E_i(x,t) + \frac{(\vec{p} \times \vec{\theta})_i}{m} \right\} = 0$$

(199)

as the differential equation satisfied by the single particle
distribution function.

Note that we have not treated the electromagnetic field
statistically. This implies an assumption as to the macro-
scopic reproducibility of $\vec{E}$, $\vec{H}$.

For our beam model we write $\omega(x, p)$ in the form

$$\omega(x, p) = \omega_0 + \delta\omega(\vec{x}, \vec{p}, t)$$

(200)

where

$$\left. \begin{array}{l} \omega_0 = n\delta(\vec{p - p_0}) \\ \\ \vec{p}_0 = m\vec{v}_0 \end{array} \right\}$$

(201)

$\vec{p}_0$ is the d-c beam electron momentum and we have assumed
that the d-c beam is uniform with electron charge density en.
In what follows we shall neglect magnetic effects. Eq. (199)
becomes

$$\frac{\partial \delta\omega}{\partial t} + \frac{\partial \delta\omega}{\partial x_\mu} \frac{p_\mu}{m} + en \frac{\partial \delta(\vec{p}-\vec{p_0})}{\partial p_\mu} E_\mu + e \frac{\partial \delta\omega}{\partial p_\mu} E_\mu = 0 \tag{202}$$

To obtain hydrodynamical equations of motion we define

$$\Delta \hat{p} = \vec{p} - \vec{p_0} \tag{203}$$

and take the zero[th] and first moments of Eq. (202) with respect to p. We make use of the following definitions for average values.

$$n + \delta n = \int [\omega_0 + \delta\omega] \, d^3 p \tag{204}$$

$$\overline{\vec{\Delta p}} = \frac{1}{(n+\delta n)} \int (\vec{p}-\vec{p_0}) \, \omega \, d^3 p \tag{205}$$

$$\overline{(\hat{\Delta p})^2} = \frac{1}{(n+\delta n)} \int (\vec{p}-\vec{p_0})^2 \, \omega \, d^3 p \tag{206}$$

$$\rho = e \int \omega \, d^3 p = e \int (\omega_0 + \delta\omega) \, d^3 p$$

$$= en + e \, \delta n \, (\vec{x}, t) \tag{207}$$

$$= \rho_0 + \rho_1$$

$$\vec{j} = \frac{e}{m}\int \omega \vec{p}\, d^3p = \frac{e}{m}\left\{ n\vec{p}_0 + \int S\omega\, \vec{p}\, d^3p \right\}$$

$$= \vec{j}_0 + \vec{j}_1 \tag{208}$$

First we multiply Eq. (202) by e and integrate over $\vec{p}$. The resulting zero[th] moment equation is

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x_\mu}\, j_{1\mu} = 0 \tag{209}$$

since the other two integrals vanish. Eq. (209) is the equation of continuity of charge for the electrons.

To calculate the first moment equation we multiply Eq. (202) by ep_ν/m and integrate over $\vec{p}$.

$$\frac{\partial}{\partial t} j_{1\nu} + \frac{\partial}{\partial x_\mu} e\int S\omega_1 \frac{p_\mu p_\nu}{m^2}\, d^3p - \frac{e^2 n}{m} E_\nu - \frac{e^2 \int Sn}{m} E_\nu = 0 \tag{210}$$

With Eq. (203) the integral term in Eq. (210) can be rewritten

$$\frac{\partial}{\partial x_\mu} \frac{e}{m^2}\left\{ p_{0\mu} p_{0\nu}\, Sn + (n+Sn)(p_{\mu 0}\overline{\Delta p_\nu} + p_{\nu 0}\overline{\Delta p_\mu}) \right.$$

$$\left. + (n+Sn)\overline{\Delta p_\mu \Delta p_\nu} \right\}. \tag{211}$$

We rewrite the first term in Eq. (210) as follows. From Eq. (208)

$$j_{1\nu} = \frac{e}{m}\int S\omega\, p_\nu\, d^3p = \frac{e}{m}\int (\omega_0+S\omega) p_\nu\, d^3p - \frac{e}{m}\int \omega_0 p_\nu\, d^3p \tag{212}$$

$$= \frac{e}{m} Sn\, p_{0\nu} + \frac{e}{m}(n+Sn)\overline{\Delta p_\nu}$$

Hence

$$\frac{\partial j_{i\nu}}{\partial t} = \frac{e}{m} (\rho_{o\nu} + \overline{\Delta p_\nu}) \frac{\partial S_\mu}{\partial t} + \frac{e}{m} (n + S_n) \frac{\partial \overline{\Delta p_\nu}}{\partial t} \tag{213}$$

but using Eqs. (209), (207) and (212) we can write

$$\frac{\partial S_\mu}{\partial t} = - \frac{1}{m} \frac{\partial}{\partial x_\mu} \left[S_n \rho_{o\nu} + (n + S_n) \overline{\Delta p_\nu} \right]. \tag{214}$$

Combining Eq. (213) and Eq. (214) we get

$$\frac{\partial j_{i\nu}}{\partial t} = - \frac{e}{m^2} (\rho_{o\nu} + \overline{\Delta p_\nu}) \frac{\partial}{\partial x_\mu} \left[S_n \rho_{o\nu} + (n + S_n) \overline{\Delta p_\nu} \right]$$

$$\qquad + \frac{e}{m} (n + S_n) \frac{\partial \overline{\Delta p_\nu}}{\partial t}. \tag{215}$$

Substituting Eqs. (211) and (215) into Eq. (210) and re-arranging terms we finally obtain

$$\left[\frac{\partial}{\partial t} + \frac{1}{m} (\rho_{o\mu} + \overline{\Delta p_\mu}) \frac{\partial}{\partial x_\mu} \right] \overline{\Delta p_\nu} - e E_\nu$$

$$= \frac{1}{m} \frac{1}{(n + S_n)} \frac{\partial}{\partial x_\mu} \left[(n + S_n) (\overline{\Delta p_\mu \Delta p_\nu} - \overline{\Delta p_\mu} \, \overline{\Delta p_\nu}) \right] \tag{216}$$

which is the hydrodynamical equation of motion. The left-hand side is the well-known equation of motion for the beam in Eulerian coordinates

$$\left[\frac{\partial}{\partial t} + \vec{\nu}(\vec{r},t)\cdot\nabla\right]\vec{\nu}(\vec{r},t) - e\vec{E} = 0$$

$$\frac{d\vec{r_0}}{dt} = 0 \qquad\qquad\qquad (217)$$

$\vec{r}$ refers to the electron position in the presence of the modulation.[16]

The term on the right-hand side of Eq. (216) is a heat diffusion term.

3. Remarks on the Large-Signal Infinite Beam Model

We specialize Eq. (216) to the case where $m\vec{\Delta}v = \vec{\Delta}p = \vec{1}_z\,\Delta p(z,t)$ and $\vec{E} = \vec{1}_z E(z,t)$. The result is

$$\left[\frac{\partial}{\partial t} + (\nu_0 + \overline{\Delta\nu})\frac{\partial}{\partial z}\right]\overline{\Delta\nu} - \frac{e}{m}E(z,t)$$

$$= \frac{1}{(n+\delta n)}\frac{\partial}{\partial z}\left\{(n+\delta n)\left[\overline{(\Delta\nu)}^2 - \overline{(\Delta\nu)^2}\right]\right\} \qquad (218)$$

We have solved Eq. (218), neglecting the right-hand side, within the framework of the infinite beam model.[17] The solution was valid up to the point where crossover sets in. Eq. (218), on the other hand, is a more general equation. It was pointed out in our discussion of the infinite beam model that at crossover the charge density becomes infinite.

16. RADC-TDR-62-199, IV Eulerian Theory, Section A, pp. 62-63.

17. Ibid, Section C, pp. 91-139.

That is, in the quasi-Eulerian formalism of our previous work the charge density, at the actual position of the electron is given by

$$\rho(\vec{3},t) = \rho(x_0 + \sigma(x_0,\theta)) = \rho_0 \left\{ \frac{1}{1 + \frac{\partial \sigma(x_0,\theta)}{\partial x_0}} - 1 \right\} \tag{219}$$

and $1 + \partial\sigma/\partial x_o$ passes through zero at crossover.

It was mentioned in connection with our previous analysis that one might question the validity of the infinite beam model at and beyond crossover. The infinite beam model assumes that one can describe an infinitely extended ion neutralized electron plasma by a set of charge sheets constrained to move in one (z) direction. The derivation of Eq. (218) has avoided the charge sheet picture and enables us to make statements regarding the validity of the infinite beam model. Eq. (218) states that the infinite beam model is a good approximation when the right-hand term is negligible. However, let us look at this term near crossover. We rewrite it in the form

$$S^{-3/2} \frac{\partial}{\partial t} \ln (n + \delta n) + \frac{\partial S}{\partial \vec{3}} \tag{220}$$

where

$$S = - \overline{(\Delta w - \overline{\Delta w})^2} \leq 0 \tag{221}$$

At crossover, according to infinite beam model theory,
$n + \delta n$ passes to infinity and it would take a peculiar can-
cellation to keep the expression (220) from becoming very
large. On the other hand, the terms on the left side of
Eq. (218) add up to zero within the framework of the infinite
beam model. We therefore conjecture that the infinite beam
model ceases to be valid at and near crossover.

IX. Infinite Beam Model With Phenomenological Losses

1. Introduction and Summary

In our statistical mechanical treatment of the infinite beam model we found that the hydrodynamical equation of motion is an approximate equation which neglects heat diffusion type terms. We attempt here to include such effects in an approximate way by the introduction of a collision loss term into the equation of motion. This device is common in plasma theory and is used to represent the effect of collisions of the electrons with neutrals. In the present situation this term should provide a step in the direction of a more realistic beam model.

In part 2 we derive an equation of motion in the presence of losses. The result is

$$\left\{ \left[\partial/\partial\theta + \partial/\partial x \right]^2 + \eta \left[\partial/\partial\theta + \partial/\partial x \right] + \lambda^2 \right\} \sigma$$

$$= -\alpha \, F_q (x + \sigma, \theta) \qquad (222)$$

The factor $2\pi\eta$ represents the number of "collisions" per r-f cycle of the modulating voltage.

In part 3 we are able to obtain a large-signal solution to Eq. (222) for the case of modulation at an infinitesimally

extended buncher gap. This solution appears in closed form and contains complete information about harmonics of all orders. The solution is

$$\sigma(x,\theta) = \frac{\sin\left\{(\lambda x)\sqrt{1-\zeta^2}\right\}}{\left\{\lambda\sqrt{1-\zeta^2}\right\}}\, e^{-\zeta(\lambda x)} \cdot \left\{\left[1+\alpha\sin(\theta-x)\right]^{1/2} - 1\right\}$$

(223)

where

$$\zeta = \frac{\eta}{2\lambda}$$

(224)

In part 4 we remark on crossover criteria in the presence of the damping term. For no losses, crossover occurs when the depth of modulation, δ, is approximately equal to 2.0. Employing the same criterion for the lossy case we find that crossover is inhibited by the damping. The critical depth of modulation moves from 2.0 in the case of no damping to the value 5.43 in the case of critical damping.

$$\zeta = \frac{\alpha\omega}{\omega_p} \qquad , \quad \alpha = \frac{V_1}{V_0}$$

$$\delta_c = 2.0 \quad \text{for} \quad \zeta = \frac{\eta}{2\lambda} = 0$$

$$\delta_c = 5.43 \quad \text{for} \quad \zeta = \frac{\eta}{2\lambda} = 1$$

(225)

2. Equation of Motion With Phenomenological Collision Losses

The infinite beam model hydrodynamical equation of motion in the absence of losses is

$$\left\{ \left[\partial/\partial_x + \partial/\partial_\theta \right]^2 + \lambda^2 \right\} \sigma = -\alpha \, F_3 (x + \sigma, \theta) \tag{226}$$

where

$$\left.\begin{array}{l}
x = \dfrac{\omega}{\nu_0} z \\[6pt]
\sigma = \dfrac{\omega}{\nu_0} \zeta \\[6pt]
\theta = \omega t \\[6pt]
\lambda = \omega_p / \omega
\end{array}\right\} \tag{227}$$

z is distance down the drift space, and ζ is the displacement caused by the presence of the modulation.

In our statistical mechanical treatment of the infinite beam model we found that Eq. (226) is an approximate equation which neglects heat diffusion terms. We shall here attempt to make a correction for these extra terms by introducing into Eq. (226) a phenomenological collision loss term. We consider that the motion is degraded by collision losses characterized by a collision frequency ν which is taken independent of velocity. Thus we enter a term in the equation of motion of the form

$$m \nu \left[\nu_0 \, \partial/\partial_z + \partial/\partial_t \right] \zeta (z, t) \tag{228}$$

Furthermore we assume that the d-c beam is produced in a way that collision losses in the unmodulated beam are accounted for, so that we are left with a beam traveling at constant velocity v_o along the z-axis.

In dimensionless units Eq. (226) now becomes

$$\left\{ \left[\frac{\partial}{\partial \theta} + \frac{1}{\lambda} \frac{\partial}{\partial x} \right]^2 + \eta \left[\frac{\partial}{\partial \theta} + \frac{1}{\lambda} \frac{\partial}{\partial x} \right] + \lambda^2 \right\} \sigma = -\alpha F_3(x + \sigma, \theta) \quad (229)$$

where

$$\left. \begin{array}{c} \eta = \nu/\omega \\ 2\pi \eta = \nu/f \end{array} \right\} \quad (230)$$

$2\pi\eta$ is the number of collisions per r-f cycle of the gap modulating field.

The device of introducing such a phenomenological collision damping term is familiar in plasma theory to account for collisions of electrons with neutral atoms and molecules. In our present application this term probably provides a step in the direction of a more realistic beam model.

3. Large-Signal Solution of the Equation of Motion

We consider the problem of modulation by an infinitesimally extended gap located at $x = 0$. The displacement σ is expressed in the form

$$\sigma(x, \theta) = \sum_{r} \sigma_r(x) e^{jr(\theta - x)} \qquad (231)$$

Substituting this expression into the homogeneous form of Eq. (229) we obtain the following equation for the $\sigma_r(x)$.

$$\left(\frac{d^2}{dx^2} + \eta \frac{d}{dx} + \lambda^2 \right) \sigma_r(x) = 0 \qquad (232)$$

We express σ_r by a Fourier integral

$$\sigma_r = \int \sigma_r(\beta) e^{-j\beta x} d\beta \qquad (233)$$

This results in the following dispersion relation.

$$\left. \begin{array}{l} \beta^2 + j\beta\eta - \lambda^2 = 0 \\ \beta = -j\varsigma\lambda \pm \lambda \sqrt{1-\varsigma^2} \\ \varsigma = \frac{\eta}{2\lambda} \end{array} \right\} \qquad (234)$$

Hence σ takes the form

$$\sigma(x, \theta) = \sum_{r} e^{jr(\theta - x) - \varsigma(\lambda x)} \left\{ \sigma_{r1} e^{j\sqrt{1-\varsigma^2}(\lambda x)} \right.$$

$$\left. + \sigma_{r2} e^{-j\sqrt{1-\varsigma^2}(\lambda x)} \right\} \qquad (235)$$

We treat the gap as a boundary condition on σ.

$$
\left.
\begin{array}{l}
\sigma(0, \theta) = 0 \\[4pt]
\left. \dfrac{\partial \sigma(x,\theta)}{\partial x} + \dfrac{\partial \sigma(x,\theta)}{\partial \theta} \right]_{x=0} = \left[(1 + \alpha \sin \theta)^{1/2} - 1 \right]
\end{array}
\right\}
\qquad (236)
$$

The first of Eqs. (236) yields

$$
\sigma_{r_1} = - \sigma_{r_2} \qquad (237)
$$

or

$$
\sigma(x, \theta) = \sum_r \sigma_r \sin\left[(\lambda x) \sqrt{1 - \xi^2} \right] e^{jr(\theta - x) - \xi(\lambda x)} \qquad (238)
$$

Making use of the second condition (236) we finally obtain

$$
\sigma(x, \theta) = \frac{\sin\left[(\lambda x) \sqrt{1 - \xi^2} \right]}{\lambda \sqrt{1 - \xi^2}} \, e^{-\xi(\lambda x)} \cdot \left\{ \left[1 + \alpha \sin(\theta - x) \right]^{1/2} - 1 \right\} \qquad (239)
$$

This solution reduces properly to the lossless case[18] when

$$
\xi = \frac{\eta}{2\lambda} \longrightarrow 0 \qquad (240)
$$

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18. RADC-TDR-62-199, IV Eulerian Theory, Section C, p. 126.

Eq. (239) exhibits a new feature, namely, that the displacement,

σ, caused by the modulation decays exponentially with x. The

1/e point is given by

$$X_d = \frac{1}{\zeta \lambda} \qquad (241)$$

4. Crossover Criteria

It is interesting to investigate the effect of the loss term,

upon crossover criteria. Crossover[19] occurs when

$$1 + \frac{\partial \sigma}{\partial x} = 0 \qquad (242)$$

From Eq. (239)

$$\frac{\partial \sigma}{\partial x} = \cos\left\{ (\lambda x) \sqrt{1-\zeta^2} \right\} e^{-\zeta(\lambda x)} \left\{ \left[1 + \alpha \sin(\theta - x) \right]^{1/2} - 1 \right\}$$

$$- (\zeta \lambda) \frac{\sin\left\{ (\lambda x)\sqrt{1-\zeta^2} \right\}}{\lambda \sqrt{1-\zeta^2}} e^{-\zeta(\lambda x)} \left\{ \left[1 + \alpha \sin(\theta - x) \right]^{1/2} - 1 \right\} \qquad (243)$$

$$- \frac{\sin\left\{ (\lambda x) \sqrt{1-\zeta^2} \right\}}{\lambda \sqrt{1-\zeta^2}} e^{-\zeta(\lambda x)} \frac{(\alpha/2) \cos(\theta - x)}{\left[1 + \alpha \sin(\theta - x) \right]^{1/2}}$$

19. RADC-TDR-62-199, IV Eulerian Theory, Section C, pp. 137-139.

To obtain some feeling for the crossover conditions we consider an electron such that

$$\theta - x = 0 \tag{244}$$

For this phase

$$\frac{\partial \sigma}{\partial x} = -(\alpha/\lambda) \frac{\sin\left\{(\lambda x)\sqrt{1-\zeta^2}\right\}}{\lambda\sqrt{1-\zeta^2}} e^{-\zeta(\lambda x)} \tag{245}$$

Crossover does not occur here if

$$I(x) = (\alpha/\lambda) \frac{\sin\left\{(\lambda x)\sqrt{1-\zeta^2}\right\}}{\lambda\sqrt{1-\zeta^2}} e^{-\zeta(\lambda x)} < 1 \tag{246}$$

The maximum of I(x) occurs at

$$x_{max} = \frac{1}{\lambda\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \tag{247}$$

Hence

$$\sin \lambda\sqrt{1-\zeta^2} \; x_{max} = \sqrt{1-\zeta^2} \tag{248}$$

and

$$I(x_{max}) = \left(\frac{\alpha}{2\lambda}\right) e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}} < 1 \tag{249}$$

We define the depth of modulation, δ, by
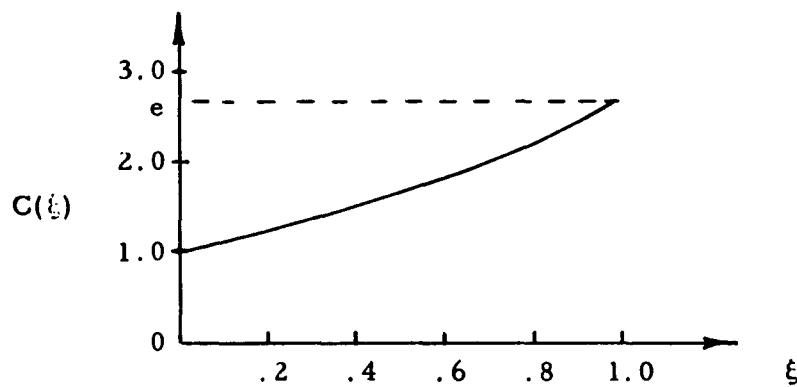
$$\zeta = \alpha/\lambda \tag{250}$$

The condition (249) becomes

$$\frac{\delta}{2} < C(\xi) \tag{251}$$

where

$$C(\xi) = e^{\frac{\xi}{\sqrt{1-\xi^2}} \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}} \tag{252}$$

$C(\xi)$ is plotted below. $\xi = 0$ corresponds to no damping and the crossover criterion (251) is

$$\delta < 2 \tag{253}$$

$\xi = 1$ corresponds to critical damping and in this case the crossover criterion becomes

$$\delta < 2 \times e = 5.43 \tag{254}$$

We see, therefore, that the presence of the collision term tends to inhibit crossover; i.e., the depth of modulation

$$\delta = \frac{\alpha\,\omega}{\omega_p} \tag{255}$$

is allowed to have larger values in the presence of damping before crossover sets in.

X. Remarks on the Possible Directions
of Future Work

We remark briefly on the directions which future work
in beam theory might take.

1. The vector normal mode formalism,[20] although
complicated, contains a fairly complete picture of the inter-
action of an electron beam with some attendant electromagnetic
structure. It provides an accurate starting point for numerical
analysis of specific devices in the large signal regime. The
analysis could profitably be extended to the case of magnetically
focused beams which allow radial motion, in particular to the
Brillouin beam.

2. We have made sizable progress in the large-signal
description of a modulated electron beam within the framework
of the infinite beam model. An accurate description has been
obtained for the case of one infinitesimally extended modulating
gap and the analysis avoids the use of perturbation theory. Some
work has been accomplished on the two-gap problem with the use
of second order perturbation theory. The multigap problem
should be analyzed in the large-signal regime by methods which
avoid perturbation theory.

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20. RADC-TDR-62-199, IV Eulerian Theory, Section B,
pp. 71-91.

Some work has been done which extends the large-signal infinite beam analysis to the case of finite geometry, specifically, the disc-model. This is a fertile area for future endeavor.

3. Our recent work with the infinite beam model indicates the possible breakdown of the hydrodynamic theory at and near crossover. Thus if one wishes to push large-signal beam theory into the region where crossover is imminent, the problem should be formulated from the point of view of statistical mechanics. The reason for this is that terms which are neglected in the hydrodynamic approximation appear to be significant in the region near crossover.

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